

VANISHING THEOREMS FOR CONSTRUCTIBLE SHEAVES ON ABELIAN VARIETIES

THOMAS KRÄMER AND RAINER WEISSAUER

ABSTRACT. We show that the cohomology of most character twists of constructible sheaves F on a complex abelian variety vanishes in degrees larger than the dimension of the support of F . For a homomorphism of abelian varieties we obtain a relative version of this generic vanishing theorem. Our proof relies on the theory of perverse sheaves and on the formalism of Tannakian categories.

INTRODUCTION

For the cohomology of coherent line bundles on a compact connected Kähler manifold Y with Albanese morphism $f : Y \rightarrow X = \text{Alb}(Y)$, one has the generic vanishing theorem of Green and Lazarsfeld [GL, th. 1] which in particular states that $H^k(Y, \mathcal{L}) = 0$ for $k < \dim(f(Y))$ and generic \mathcal{L} in $\text{Pic}^0(Y)$. We give a more general vanishing theorem for constructible sheaves on X which contains the above theorem as a special case. In the context of constructible sheaves, the role of the coherent line bundles \mathcal{L} is taken over by the local systems L_χ attached to characters $\chi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$ of the fundamental group $\pi_1(X, 0)$. Our theorem easily implies a relative generic vanishing theorem for a homomorphism between abelian varieties as we show in section 1. In section 2 we illustrate how the Kodaira-Nakano type theorems of loc. cit. can be recovered from our results. Our methods also allow to study the precise locus of characters χ for which the vanishing theorem fails, see section 13.

We consider \mathbb{C} -sheaves that are constructible with respect to an analytic stratification. On complex projective varieties, constructibility with respect to an analytic stratification is of course equivalent to constructibility with respect to an algebraic stratification by Chow's theorem.

Theorem 1. *Let F be a constructible sheaf on a complex abelian variety X . Then for most characters χ we have*

$$H^i(X, F \otimes_{\mathbb{C}} L_\chi) = 0 \quad \text{for } i > \dim(\text{Supp } F).$$

Here we use the following terminology: For abelian subvarieties $A \subseteq X$ let $K(A)$ denote the group of characters $\chi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$ whose restriction to the subgroup $\pi_1(A, 0)$ of $\pi_1(X, 0)$ is trivial. We then say that a statement

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holds for most characters χ if it holds for all χ in the complement of a thin set, where by a thin set of characters we mean a finite union of translates $\chi_i \cdot K(A_i)$ for certain non-zero abelian subvarieties $A_i \subseteq X$ and certain characters χ_i of $\pi_1(X, 0)$. The same terminology will be used in theorems 4 and 5 for line bundles $\mathcal{L} \in \text{Pic}^0(X)$.

For a complex analytic space Y we have the derived category $D_c^b(Y, \mathbb{C})$ of bounded \mathbb{C} -sheaf complexes with constructible cohomology sheaves. The perverse cohomology sheaves ${}^p H^n(K) = {}^p \tau_{\leq n} {}^p \tau_{\geq n}(K)$ of $K \in D_c^b(Y, \mathbb{C})$ are defined by the truncation functors ${}^p \tau_{\leq n}$ and ${}^p \tau_{\geq n}$ of the perverse t -structure on $D_c^b(Y, \mathbb{C})$. The core of this t -structure is the abelian category $\text{Perv}(Y, \mathbb{C})$ of perverse sheaves on Y . Recall that a complex $K \in D_c^b(Y, \mathbb{C})$ is said to be semi-perverse if its cohomology sheaves $\mathcal{H}^{-i}(K)$ satisfy the estimate $\dim(\text{Supp } \mathcal{H}^{-i}(K)) \leq i$ for all $i \in \mathbb{Z}$, and that K is a perverse sheaf iff K and its Verdier dual DK are both semi-perverse. For our proof of theorem 1 the following reformulation will be crucial.

Theorem 2. *Let P be a perverse sheaf on a complex abelian variety X . Then for most χ we have*

$$H^i(X, P \otimes_{\mathbb{C}} L_{\chi}) = 0 \quad \text{for } i \neq 0.$$

Dévissage with respect to the perverse t -structure and Verdier duality show that theorem 2 is equivalent to the statement that for any semi-perverse complex P on X , one has $H^i(X, P \otimes_{\mathbb{C}} L_{\chi}) = 0$ for $i > 0$ and most χ . So theorem 2 implies theorem 1, because the complex $P = F[\dim(\text{Supp } F)]$ is semi-perverse for any constructible sheaf F . Conversely, theorem 1 implies theorem 2 by dévissage with respect to the standard t -structure.

Our proof of theorem 2 will occupy sections 3 through 9 and is based on two ingredients. One of them is of a group-theoretic nature and depends crucially on a result of Deligne [De2]. Using the decomposition theorem, we construct from semisimple perverse sheaves on X a semisimple abelian rigid symmetric monoidal category in the spirit of [We2]. By [De2] this is a Tannakian category, so we can apply arguments from group theory to reduce the proof of our theorem to the second main ingredient, the classification of simple perverse sheaves with Euler characteristic zero. Since the Tannakian aspect of the proof is of independent interest, we explain in sections 10 through 12 how it can be extended to the non-semisimple case. For further applications we survey the main properties of the arising Tannaka groups.

1. A RELATIVE GENERIC VANISHING THEOREM

Let X be a complex abelian variety and $A \subseteq X$ an abelian subvariety with quotient morphism $f : X \rightarrow B = X/A$. Assuming theorem 2 for the abelian variety A , we obtain the following relative generic vanishing theorem; here the quantifier *most* can be read in the slightly stronger sense that it does not refer to the characters of $\pi_1(X, 0)$ but rather to their pull-back to the

subgroup $\pi_1(A, 0) \subseteq \pi_1(X, 0)$, see section 13 and in particular the remark preceding lemma 35.

Theorem 3. *Let P be a perverse sheaf on X . Then for most characters χ the direct image $Rf_*(P \otimes_{\mathbb{C}} L_{\chi})$ is a perverse sheaf on B .*

Proof. Put $P_{\chi} = P \otimes_{\mathbb{C}} L_{\chi}$. By Verdier duality it will be enough to show that for most characters χ the direct image complex $Rf_*(P_{\chi})$ satisfies the semi-perversity condition

$$\dim(\text{Supp } \mathcal{H}^{-k}(Rf_*(P_{\chi}))) \leq k \quad \text{for all } k \in \mathbb{Z}.$$

To check this condition, note that by lemma 2.4 and section 3.1 in [BF] we can find Whitney stratifications $X = \sqcup_{\beta} X_{\beta}$ and $B = \sqcup_{\alpha} B_{\alpha}$ such that

- a) the cohomology sheaves $\mathcal{H}^{-i}(P_{\chi}) = \mathcal{H}^{-i}(P) \otimes_{\mathbb{C}} L_{\chi}$ are locally constant on the strata X_{β} for all β, i and χ ,
- b) each $f(X_{\beta})$ is contained in some B_{α} , and
- c) for all α, β with $f(X_{\beta}) \subseteq B_{\alpha}$ the restriction $f : X_{\beta} \rightarrow B_{\alpha}$ is smooth.

By theorem 4.1 of loc. cit. then the restriction $\mathcal{H}^{-k}(Rf_*(P_{\chi}))|_{B_{\alpha}}$ is locally constant for all α, k and χ . Since there are only finitely many strata B_{α} and since $\mathcal{H}^{-k}(Rf_*(P_{\chi})) \neq 0$ for only finitely many k , it follows that if the direct image complex $Rf_*(P_{\chi})$ were not semi-perverse for most χ , then we could find α and k such that

- d) $\dim(B_{\alpha}) > k$ (where as usual by the dimension of a constructible subset we mean the maximum of the dimensions of the irreducible components of its closure), and
- e) $\mathcal{H}^{-k}(Rf_*(P_{\chi}))_b \neq 0$ for all points $b \in B_{\alpha}(\mathbb{C})$ and all χ in a set of characters which is not thin in the sense of the introduction.

Indeed, if a property does not hold for most characters, then by definition it fails on a set of characters which is not thin. Fixing α and k as above, we now argue by contradiction.

Fix $b \in B_{\alpha}(\mathbb{C})$. Consider the fibre $F_b = f^{-1}(b)$, and for arbitrary χ denote by $M_b = P_{\chi}|_{F_b}$ the restriction of P_{χ} to F_b (we suppress the character twist in this notation). Put

$$M_b^r = {}^p H^{-r}(M_b) \quad \text{for } r \in \mathbb{Z},$$

and consider the spectral sequence

$$E_2^{rs} = H^{-s}(F_b, M_b^r) \implies H^{-(r+s)}(F_b, M_b) = \mathcal{H}^{-(r+s)}(Rf_*(P_{\chi}))_b.$$

Theorem 2 for $F_b \cong A$ shows that for most χ we have $H^{-s}(F_b, M_b^r) = 0$ for all $s \neq 0$ and all $r \in \mathbb{Z}$. For such χ the spectral sequence degenerates, i.e.

$$\mathcal{H}^{-k}(Rf_*(P_{\chi}))_b = H^0(F_b, M_b^k).$$

On the other hand, by e) we can assume $\mathcal{H}^{-k}(Rf_*(P_{\chi}))_b \neq 0$. By the above then $M_b^k \neq 0$. Since $M_b^k = {}^p H^0(M_b[-k])$, it follows by definition of the

perverse t -structure that

$$\dim(\mathrm{Supp} \mathcal{H}^{-i}(M_b)) = i - k \geq 0 \quad \text{for some } i \in \mathbb{Z}.$$

Now by a) the support of $\mathcal{H}^{-i}(P_\chi)$ is a union of certain strata X_β , so using the above dimension estimate and the definition of $M_b = P_\chi|_{F_b}$ we find a stratum $X_\beta \subseteq \mathrm{Supp} \mathcal{H}^{-i}(P_\chi)$ with $\dim(F_b \cap X_\beta) = i - k$. Since by b) and c) the stratum X_β is equidimensional over B_α , it follows that

$$\dim(\mathrm{Supp} \mathcal{H}^{-i}(P_\chi)) \geq \dim(X_\beta) = i - k + \dim(B_\alpha).$$

But $\dim(B_\alpha) > k$ by property d), so it follows that the perverse sheaf P_χ is not semi-perverse, a contradiction. \square

Note that in the proof of theorem 3 we have only used theorem 2 for the fibres $f^{-1}(b) \cong A$ but not for X itself. Indeed, using this observation and assuming theorem 2 only for simple abelian varieties, by induction on the dimension one can deduce for arbitrary abelian varieties a slightly weaker version of theorem 2 where *most* is replaced by *generic* [We4].

2. KODAIRA-NAKANO-TYPE VANISHING THEOREMS

From theorem 2 one easily recovers stronger versions of the vanishing theorems of Green and Lazarsfeld as follows. Let Y be a compact connected Kähler manifold of dimension d . By a local system on Y we mean a locally constant sheaf of complex vector spaces. Let

$$f : Y \rightarrow X = \mathrm{Alb}(Y)$$

be the Albanese morphism. Put $X_n = \{x \in X \mid \dim(f^{-1}(x)) = n\}$ for $n \in \mathbb{N}_0$ and define $w(Y) = \min\{2d - (\dim(X_n) + 2n) \mid n \in \mathbb{N}_0, X_n \neq \emptyset\}$. Notice that $w(Y) \leq d$. Indeed, for some n the preimage $f^{-1}(X_n)$ is dense in Y so that $d = \dim(f^{-1}(X_n)) = \dim(X_n) + n$, hence $2d - (\dim(X_n) + 2n)$ is equal to $2d - (d + n) = d - n \leq d$.

In particular, the morphism f is semi-small in the sense of [KW, III.7] if and only if $w(Y) = d$. Furthermore, for local systems E on Y one sees as in loc. cit. that the perverse sheaf $E[d]$ satisfies ${}^p H^n(Rf_*(E[d])) = 0$ for all $n > d - w(Y)$, so

$$Rf_* E[2d - w(Y)] \quad \text{is semi-perverse.}$$

Hence theorem 2 implies the following version of [GL, th. 2].

Theorem 4. *Let E be a local system on Y . Then for most $\mathcal{L} \in \mathrm{Pic}^0(Y)$,*

$$H^p(Y, \Omega_Y^q(E \otimes_{\mathbb{C}} \mathcal{L})) = 0 \quad \text{for } p + q < w(Y).$$

Proof. $f^* : \mathrm{Pic}^0(X) \rightarrow \mathrm{Pic}^0(Y)$ is an isomorphism [GH, p. 553], so every line bundle $\mathcal{L} \in \mathrm{Pic}^0(Y)$ is the pull-back of some $\mathcal{M} \in \mathrm{Pic}^0(X)$. Since \mathcal{M}

admits a flat connection, we have $\mathcal{M} = \mathcal{O}_X \otimes_{\mathbb{C}} L_{\chi}$ for some character χ . Hence $\mathcal{L} = f^*(\mathcal{M}) = \mathcal{O}_Y \otimes_{\mathbb{C}} f^*(L_{\chi})$, and by Hodge theory

$$\bigoplus_{p+q=k} H^p(Y, \Omega_Y^q(E \otimes_{\mathbb{C}} \mathcal{L})) = H^k(Y, E \otimes_{\mathbb{C}} f^*(L_{\chi})).$$

The right hand side is equal to the hypercohomology group

$$H^k(X, Rf_* E \otimes_{\mathbb{C}} L_{\chi}) = H^{k-2d+w(Y)}(X, Rf_* E[2d-w(Y)] \otimes_{\mathbb{C}} L_{\chi}).$$

Since $Rf_* E[2d-w(Y)] \otimes_{\mathbb{C}} L_{\chi}$ is semi-perverse, theorem 2 shows that the above hypercohomology group vanishes for most χ if $k > 2d-w(Y)$. The theorem now follows by Serre duality. \square

For $n \in \mathbb{N}_0$ consider $\bar{X}_n = \{x \in X \mid \dim(f^{-1}(x)) \geq n\}$ and $\bar{Y}_n = f^{-1}(\bar{X}_n)$, and put $d_n = \dim(\bar{Y}_n)$ with $d_n = -\infty$ for $\bar{Y}_n = \emptyset$.

Theorem 5. *Suppose $p+q = d-n$ for some $n \geq 1$. Then*

$$H^p(Y, \Omega_Y^q(\mathcal{L})) = 0$$

holds for most \mathcal{L} in $\text{Pic}^0(Y)$ unless $d - d_n \leq p, q \leq d_n - n$.

Proof. Since Y is Kähler and f is proper, by the decomposition theorem [Sa3, th. 0.6] we have $Rf_* \mathbb{C}_Y[d] \cong \bigoplus_m M_m[-m]$ for certain pure Hodge modules M_m on X of weight $m+d$ in the sense of [Sa1]. For characters χ and $K \in D_c^b(X, \mathbb{C})$ put $K_{\chi} = K \otimes_{\mathbb{C}} L_{\chi}$. Then by theorem 2

$$H^{d-n}(Y, f^*(L_{\chi})) = H^{-n}(X, (Rf_* \mathbb{C}_Y[d])_{\chi}) \cong H^0(X, (M_{-n})_{\chi})$$

for most χ . Since $(M_{-n})_{\chi}$ is a pure Hodge module of weight $p+q$, the cohomology group

$$H^{d-n}(Y, f^*(L_{\chi})) \cong H^0(X, (M_{-n})_{\chi})$$

carries a pure Hodge structure of weight $p+q$. We are looking for bounds on the types (p, q) occuring in this Hodge structure. Equivalently, we must find bounds on the types $(d-q, d-p)$ occuring in the pure Hodge structure of weight $d+n$ on the dual

$$H^{d+n}(Y, f^*(L_{\chi^{-1}})) \cong H^0(X, (M_{-n})_{\chi}^*(-d).$$

The Verdier dual $D(\mathbb{C}_Y[d])$ is isomorphic to \mathbb{C}_Yd, so by properness of f we have $D(M_{-n}) \cong M_n(d)$. Hence

$$H^0(X, (M_{-n})_{\chi}^*(-d) \cong H^0(X, (M_n)_{\chi^{-1}}).$$

It is easy to see that $\text{Supp}(M_n) = \text{Supp}(M_{-n}) \subseteq \bar{X}_n$. Thus by base change $M_n[-n]$ is a direct summand of $Rf_* \mathbb{C}_{\bar{Y}_n}[d]$.

Now choose a composition $\pi : \tilde{Y} \rightarrow Y$ of blow-ups in smooth centers defining an embedded resolution of singularities $\tilde{Y}_n = \pi^{-1}(\bar{Y}_n) \rightarrow \bar{Y}_n$, see [Hi] or [BM, th. 10.7]. Then $\mathbb{C}_Y[d]$ is a direct summand of $R\pi_* \mathbb{C}_{\tilde{Y}}[d]$ due to the decomposition theorem for projective morphisms [Sa2, cor. 3, p. 857].

Hence $\mathbb{C}_{\bar{Y}_n}[d]$ is a direct summand of $R\pi_*\mathbb{C}_{\tilde{Y}_n}[d]$. Thus $M_n[-n]$ is a direct summand of $Rf_*R\pi_*\mathbb{C}_{\tilde{Y}_n}[d]$. So we get a monomorphism

$$H^0(X, (M_n)_{\chi^{-1}}) \hookrightarrow H^{d+n}(\tilde{Y}_n, \pi^* f^*(L_{\chi^{-1}}))$$

compatible with the Hodge structures on both sides. But for the Hodge types $(d-p, d-q)$ of the pure Hodge structure on $H^{d+n}(\tilde{Y}_n, \pi^* f^*(L_{\chi})^{-1})$ we have the inequality $\max(d-p, d-q) \leq d_n = \dim(\tilde{Y}_n)$, and using that $p+q = d-n$ we then also get $\min(d-p, d-q) \geq d+n-d_n$. Hence $\max(p, q) \leq d_n - n$ and $\min(p, q) \geq d - d_n$. \square

This includes the generic vanishing theorem of Green and Lazarsfeld quoted in the introduction. Indeed, $p+q = d-n < \dim(f(Y))$ implies that n is larger than the dimension $d - \dim(f(Y))$ of the generic fiber; so $d_n < d$ and hence $\min(p, q) \geq 1$ for most \mathcal{L} . If Y is algebraic, theorem 5 holds also for $H^p(Y, \Omega_Y^q(E \otimes_{\mathbb{C}} \mathcal{L}))$ with a local coefficient system E on Y .

In general the above bounds are strict: If $d = 4$ and if Y is the blow-up of X along a smooth algebraic curve $C \subset X$ of genus ≥ 2 , then $w(Y) = d_1 = 3$ but $H^2(Y, \Omega_Y^1(\mathcal{L})) \neq 0$ for all non-trivial \mathcal{L} , see [GL, top of p. 402].

3. THE SETTING

Let X be an abelian variety over an algebraically closed field k which has characteristic zero or is the algebraic closure of a finite field. As in [BBD] we denote by $D_c^b(X, \Lambda)$ the triangulated category of complexes of Λ -sheaves with bounded constructible cohomology sheaves, where Λ is a subfield of \mathbb{C} if $\text{char}(k) = 0$, resp. a subfield of $\overline{\mathbb{Q}}_l$ for some prime $l \neq \text{char}(k)$ otherwise. Let $\pi_1(X, 0)$ be the topological fundamental group if $\Lambda \subseteq \mathbb{C}$, resp. the étale fundamental group if $\Lambda \subseteq \overline{\mathbb{Q}}_l$. In the latter case, by a character we always mean a continuous character with image in a finite extension field of \mathbb{Q}_l .

Let $a : X \times X \rightarrow X$ denote the group law. Then $D_c^b(X, \Lambda)$ is a Λ -linear rigid symmetric monoidal category with respect to the convolution product $*$: $D_c^b(X, \Lambda) \times D_c^b(X, \Lambda) \rightarrow D_c^b(X, \Lambda)$ defined by

$$K_1 * K_2 = Ra_*(K_1 \boxtimes K_2);$$

see [We1, sect. 2.1] and [We5]. The rigid dual of an object K in $D_c^b(X, \Lambda)$ is given in terms of its Verdier dual DK by

$$K^\vee = (-id_X)^* DK,$$

and the unit object $\mathbf{1}$ of $D_c^b(X, \Lambda)$ is the skyscraper sheaf δ_0 of rank one with support in the origin. Every skyscraper sheaf $K = \delta_x$ of rank one, supported in a point $x \in X(\mathbb{C})$, is an invertible object in the sense that the evaluation map $K^\vee * K \rightarrow \mathbf{1}$ is an isomorphism. For $\text{char}(k) = 0$ every invertible object has this form, see proposition 21(b). If we want to stress the symmetric monoidal category structure on $D_c^b(X, \Lambda)$, we write $(D_c^b(X, \Lambda), *)$.

For the rest of this paper it will be convenient to work in a flexible axiomatic setting. Let $(\mathbf{D}, *)$ be a Λ -linear rigid symmetric monoidal category with $\text{End}_{\mathbf{D}}(\mathbf{1}) \cong \Lambda$ together with a faithful Λ -linear tensor functor ACU

$$\text{rat} : (\mathbf{D}, *) \rightarrow (D_c^b(X, \Lambda), *) .$$

The notation rat is motivated by the case where $k = \mathbb{C}$ and $\Lambda = \mathbb{Q}$, and where $\mathbf{D} = D^b(\text{MHM}(X))$ is the bounded derived category of the category $\text{MHM}(X)$ of mixed Hodge modules in the sense of [Sa1].

For $K \in \mathbf{D}$ we denote the hypercohomology resp. the Euler characteristic of $\text{rat}(K)$ simply by $H^\bullet(X, K)$ resp. by $\chi(K)$. Depending on the context we require some of the following properties.

- (1) *Degree shifts.* We have an auto-equivalence $K \mapsto K[1]$ on \mathbf{D} which induces the usual shift functor on $D_c^b(X, \Lambda)$.
- (2) *Perverse truncations.* We have endofunctors ${}^p\tau_{\leq n}, {}^p\tau_{\geq n}$ on \mathbf{D} and natural transformations ${}^p\tau_{\leq n} \rightarrow \text{id}_{\mathbf{D}}, \text{id}_{\mathbf{D}} \rightarrow {}^p\tau_{\geq n}$, which induce on $D_c^b(X, \Lambda)$ the perverse truncations.
Furthermore, the essential image \mathbf{P} of the perverse cohomology functors ${}^pH^n = {}^p\tau_{\leq n} \circ {}^p\tau_{\geq n}$ is an *abelian* full subcategory of \mathbf{D} , and the functor $\text{rat} : \mathbf{P} \rightarrow \text{Perv}(X, \Lambda)$ is exact.
- (3) *Character twists.* For any character χ of $\pi_1(X, 0)$ we are given an endofunctor $K \mapsto K_\chi$ on \mathbf{D} , which induces on $D_c^b(X, \Lambda)$ the functor $K \mapsto K \otimes_\Lambda L_\chi$ for the local system L_χ attached to χ .
- (4) *Perverse decomposition.* For all $K \in \mathbf{D}$ we have a (non-canonical) isomorphism $K \cong \bigoplus_{n \in \mathbb{Z}} {}^pH^{-n}(K)[n]$.
- (5) *Semisimplicity.* In (2) the abelian category \mathbf{P} is semisimple.
- (6) *Hard Lefschetz.* In \mathbf{D} there exists an invertible object $\mathbf{1}(1)$ whose image in $\text{Perv}(X, \Lambda)$ under rat is the Tate twist of $\mathbf{1}$. For all K, L in \mathbf{D} and all $i \geq 0$ we have functorial Lefschetz isomorphisms

$${}^pH^{-i}(K * L) \cong {}^pH^i(K * L)(i) ,$$

where the Tate twist (i) means i -fold convolution with $\mathbf{1}(1)$.

Note that we do not assume \mathbf{D} to be triangulated. Indeed we will later consider the following non-triangulated categories.

Example 6. *The axioms (1)-(6) hold for the full subcategory $\mathbf{D} \subseteq D_c^b(X, \Lambda)$ consisting of all direct sums of degree shifts of semisimple perverse sheaves which are defined over some finite field in case $\text{char}(k) > 0$.*

For $k = \mathbb{C}$ this follows from [Dr] together with [BK] or [Gai]; it also follows from [Sab] and [Mo]. For $\text{char}(k) > 0$ one can invoke the mixedness results of [Laf] and [BBD]. In example 6 we could also replace \mathbf{D} by the full subcategory of objects of geometric origin in the sense of [BBD, p. 163].

Example 7. *Axioms (1)-(6) hold for $k = \mathbb{C}$ if $\mathbf{D} \subseteq D^b(\text{MHM}(X)) \otimes \Lambda$ is the full subcategory of all direct sums of degree shifts of semisimple Hodge modules, with coefficients extended from \mathbb{Q} to Λ .*

In the above axiomatic setting we will consider full subcategories \mathbf{N} of \mathbf{D} which consist of objects that are negligible for our purposes. For these we consider the following properties:

- (N1) *Stability.* $\mathbf{N} * \mathbf{D} \subseteq \mathbf{N}$, and \mathbf{N} is stable under taking retracts, degree shifts, perverse truncations and rigid duals.
- (N2) *Twist acyclicity.* Every $K \in \mathbf{N}$ satisfies $H^\bullet(X, K_\chi) = 0$ for most χ .
- (N3) *Acyclic objects.* \mathbf{N} contains all $K \in \mathbf{D}$ with $H^\bullet(X, K) = 0$.
- (N4) *Trace zero objects.* \mathbf{N} contains all simple objects of \mathbf{P} with vanishing Euler characteristic.

Remark 8. *For a set Γ of characters let $\mathbf{N} \subseteq \mathbf{D}$ be the full subcategory of all $K \in \mathbf{D}$ such that $\text{rat}(K)$ is a direct sum of degree shifts of local systems L_χ with $\chi \in \Gamma$. Then axioms (N1) and (N2) are satisfied for \mathbf{N} .*

Proof. $L_\chi * K = L_\chi \otimes H^\bullet(X, K(\chi^{-1}))$ by [We1, p. 20], so $\mathbf{N} * \mathbf{D} \subseteq \mathbf{N}$ and axiom (N1) holds. For (N2) use that $H^\bullet(X, L_\chi) = 0$ iff χ is non-trivial. \square

4. CHARACTER TWISTS AND CONVOLUTION

In this section we assume that the Λ -linear rigid symmetric monoidal category \mathbf{D} and the faithful tensor functor $\text{rat} : \mathbf{D} \rightarrow D_c^b(X, \Lambda)$ satisfy axioms (1)-(3) of section 3.

Proposition 9. *For any character χ , the auto-equivalence $K \mapsto K_\chi$ of \mathbf{D} defines a tensor functor ACU compatible with degree shifts and perverse truncations.*

Proof. By faithfulness of rat we may assume $\mathbf{D} = D_c^b(X, \Lambda)$ and $\text{rat} = \text{id}$. The functor $K \mapsto K_\chi = K \otimes_\Lambda L_\chi$ preserves semi-perversity, so it is t -exact with respect to the perverse t -structure since $D(K_\chi) \cong D(K)_{\chi^{-1}}$. It remains to check tensor functoriality. Clearly $\mathbf{1}_\chi = \mathbf{1}$.

Depending on the context, put $R = \mathbb{Z}_l$, $R = \mathbb{Z}$ or $R = \mathbb{Z}/n\mathbb{Z}$ (the case where $p = \text{char}(k)$ divides n is included). The group law $a : X \times X \rightarrow X$ induces on cohomology the diagonal map

$$a^* : H^1(X, R) \rightarrow H^1(X \times X, R) = H^1(X, R) \oplus H^1(X, R), x \mapsto (x, x).$$

In the first two cases use the formula preceding lemma 15.2 in [Mi1]. In the last case notice $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$ for $(n, p) = 1$, since k is algebraically closed, and $H^1(X, \mu_n) \cong \text{Pic}^0(X)[n]$ by [Mi2, cor. III.4.18]. Thus the claim follows, since $a^*(\mathcal{L}) \cong \text{pr}_1^*(\mathcal{L}) \otimes \text{pr}_2^*(\mathcal{L})$ holds for line bundles $\mathcal{L} \in \text{Pic}^0(X)$; see [Mi1, prop. 9.2]. On the other hand $H^1(X, \mathbb{Z}/n\mathbb{Z}) \cong H^1(X, W_r)^F$ for $n = p^r$ by [S1, prop. 13]. In this case, by taking Frobenius invariants, the result follows from $H^1(X \times X, W_r) = H^1(X, W_r) \oplus H^1(X, W_r)$; see [S2, p. 136].

Now

$$H^1(X, R) = \text{Hom}(\pi_1(X, 0), R),$$

where in the étale setting we require the homomorphisms to be continuous; see [Mi1, rem. 15.5] for $R = \mathbb{Z}_l$ and [S1, p. 50] for $R = \mathbb{Z}/n\mathbb{Z}$. If we write the group structure on fundamental groups additively, it follows that

$$a_* : \pi_1(X, 0) \times \pi_1(X, 0) = \pi_1(X \times X, 0) \rightarrow \pi_1(X, 0)$$

is the addition morphism $(x, y) \mapsto x + y$. For $\psi \in \text{Hom}(\pi_1(X, 0), R)$ this implies $\psi(a_*(x, y)) = \psi(x + y) = \psi(x) + \psi(y)$, i.e. $\psi \circ a_* = \psi \boxtimes \psi$ as an additive character on $\pi_1(X, 0) \times \pi_1(X, 0) = \pi_1(X \times X, 0)$. For multiplicative characters $\chi : \pi_1(X, 0) \rightarrow \Lambda^*$ this implies

$$\chi(a_*(x, y)) = \chi(x + y) = \chi(x) \cdot \chi(y), \quad \text{i.e.} \quad \chi \circ a_* = \chi \boxtimes \chi.$$

Indeed, for $\Lambda \subseteq \mathbb{C}$ one has $\text{Hom}(\pi_1(X, 0), R) \otimes_R \mathbb{C}^* = \text{Hom}(\pi_1(X, 0), \mathbb{C}^*)$ taking $R = \mathbb{Z}$. For $\Lambda \subseteq \overline{\mathbb{Q}_l}$ any multiplicative character χ takes values in $E^* \cong \mathbb{Z} \times F^* \times U$, where F is the residue field of a finite extension field E of \mathbb{Q}_l and U is its group of 1-units. By continuity, $\chi = \chi_F \cdot \chi_U$ for characters χ_F and χ_U with values in F^* resp. U . The character χ_U can be handled as above, and the discussion for the character χ_F is covered by the case $R = \mathbb{Z}/n\mathbb{Z}$ with $n = \#F^*$.

For the local system $L = L_\chi$ defined by a character $\chi : \pi_1(X, 0) \rightarrow \Lambda^*$, this gives an isomorphism on $X \times X$

$$\varphi : a^*L \xrightarrow{\sim} L \boxtimes L.$$

Note that φ is uniquely determined up to multiplication by an element of Λ^* . In what follows, we fix a choice of φ once and for all. The choice of φ will not matter for the commutativity of the diagrams below, as long as we use the same φ consistently. However, since a tensor functor is not determined by the underlying functor alone, different choices of φ give different (but isomorphic) tensor functors. For us, it is most convenient to fix a trivialization $\lambda : L_0 \cong \Lambda$ of the stalk L_0 at the origin 0 of X , and to require that the stalk morphism $\varphi_0 : a^*L_{(0,0)} \rightarrow (L \boxtimes L)_{(0,0)} = L_0 \otimes_\Lambda L_0$ at the origin $(0, 0)$ of $X \times X$ makes the following diagram commutative:

$$\begin{array}{ccc} (a^*L)_{(0,0)} & \xrightarrow{\varphi_0} & L_0 \otimes_\Lambda L_0 \\ \parallel & & \downarrow \lambda \otimes \lambda \\ L_0 & \xrightarrow{\lambda} & \Lambda \end{array}$$

Here $L_0 = e_X^*(L) = e_{X^2}^*a^*(L) = (a^*L)_{(0,0)}$ since $a \circ e_{X^2} = e_X$ holds for the unit sections $e_X : \{0\} \rightarrow X$ and $e_{X^2} : \{(0, 0)\} \rightarrow X^2$. For the unique $v \in L_0$ such that $\lambda(v) = 1$, we have $\varphi_0^{-1}(\alpha \cdot v \otimes \beta \cdot v) = \alpha\beta \cdot v$ for $\alpha, \beta \in \Lambda$.

Let $A, B \in D_c^b(X, \Lambda)$, and let $p_1, p_2 : X \times X \rightarrow X$ be the projections onto the two factors. Using our fixed choice of φ , we get an isomorphism

$$\psi : (A * B)_\chi \xrightarrow{\sim} A_\chi * B_\chi$$

defined by the commutative diagram

$$\begin{array}{ccc}
 (A * B)_\chi & \xrightarrow{\quad \psi \quad} & A_\chi * B_\chi \\
 \parallel & & \parallel \\
 (Ra_*(A \boxtimes B)) \otimes L & & Ra_*((A \otimes L) \boxtimes (B \otimes L)) \\
 \parallel & & \cong \uparrow Ra_*(id \otimes S' \otimes id) \\
 Ra_*((A \boxtimes B) \otimes a^*L) & \xrightarrow{Ra_*(id \otimes \varphi)} & Ra_*((A \boxtimes B) \otimes (L \boxtimes L))
 \end{array}$$

where by $S' : p_2^*(B) \otimes p_1^*(L) \xrightarrow{\sim} p_1^*(L) \otimes p_2^*(B)$ we denote the symmetry constraint of the tensor product.

The isomorphisms ψ are compatible with the symmetry constraint S of the symmetric monoidal category $(D_c^b(X, \Lambda), *)$, i.e. for all A, B in $D_c^b(X, \Lambda)$ the diagram

$$\begin{array}{ccc}
 (A * B)_\chi & \xrightarrow{\quad \psi \quad} & A_\chi * B_\chi \\
 S_\chi \downarrow & & \downarrow S \\
 (B * A)_\chi & \xrightarrow{\quad \psi \quad} & B_\chi * A_\chi
 \end{array}$$

is commutative. Indeed, unravelling the definitions, the commutativity of the above diagram is equivalent to the commutativity of the diagram

$$\begin{array}{ccc}
 a^*L & \xrightarrow{\quad \varphi \quad} & L \boxtimes L = p_1^*L \otimes p_2^*L \\
 \parallel & & \cong \uparrow S' \\
 \sigma^*a^*L & \xrightarrow{\quad \sigma^*(\varphi) \quad} & \sigma^*(L \boxtimes L) = p_2^*L \otimes p_1^*L
 \end{array}$$

where $\sigma : X \times X \rightarrow X \times X$ is the morphism $(x, y) \mapsto (y, x)$ and S' is the symmetry constraint of the tensor product. Since our diagram commutes up to a scalar in Λ^* , it suffices to check commutativity on the stalks at $(0, 0)$. This boils down to the property $(\lambda \otimes \lambda)(u \otimes v) = (\lambda \otimes \lambda)(v \otimes u)$ of the rigidification λ .

The isomorphisms ψ are also compatible with the associativity constraint of the symmetric monoidal category $(D_c^b(X, \Lambda), *)$. Indeed, by strictness [We1, p. 11], the associativity constraints are the identity morphisms, so it suffices that the diagram

$$\begin{array}{ccccc}
 ((A * B) * C)_\chi & \xrightarrow{\quad \psi \quad} & ((A * B)_\chi) * C_\chi & \xrightarrow{\quad \psi * id \quad} & (A_\chi * B_\chi) * C_\chi \\
 \parallel & & & & \parallel \\
 (A * (B * C))_\chi & \xrightarrow{\quad \psi \quad} & A_\chi * ((B * C)_\chi) & \xrightarrow{\quad id * \psi \quad} & A_\chi * (B_\chi * C_\chi)
 \end{array}$$

commutes for all $A, B, C \in D_c^b(X, \Lambda)$. Writing

$$((A * B) * C)_\chi = Ra_*R(a \times id)_*((A \boxtimes B) \boxtimes C) \otimes (a \times id)^*a^*L$$

and similarly for the other convolutions, the commutativity of the diagram becomes equivalent to the commutativity of the diagram on $X \times X \times X$

$$\begin{array}{ccccc}
 (a \times id)^* a^* L & \xrightarrow{(a \times id)^* \varphi} & (a \times id)^* L \boxtimes L = a^* L \boxtimes L & \xrightarrow{\varphi \boxtimes id} & (L \boxtimes L) \boxtimes L \\
 \parallel & & & & \parallel \\
 (id \times a)^* a^* L & \xrightarrow{(id \times a)^* \varphi} & (id \times a)^* L \boxtimes L = L \boxtimes a^* L & \xrightarrow{id \boxtimes \varphi} & L \boxtimes (L \boxtimes L)
 \end{array}$$

Again it suffices to check the commutativity on stalks at $(0, 0, 0)$. The upper arrow becomes the composition

$$(\varphi \otimes id) \circ \varphi : L_0 \rightarrow L_0 \otimes_{\Lambda} L_0 \rightarrow (L_0 \otimes_{\Lambda} L_0) \otimes_{\Lambda} L_0.$$

Its inverse maps $(\alpha \cdot v \otimes \beta \cdot v) \otimes \gamma \cdot v$ to $(\alpha\beta)\gamma \cdot v$. By a similar computation for the lower row, the commutativity of the diagram hence boils down to the associativity law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ of the field Λ . \square

Corollary 10. *For $K \in \mathbf{D}$ the Euler characteristic of K_{χ} does not depend on the character χ .*

Proof. Replacing K by $rat(K)$, we can assume that $\mathbf{D} = D_c^b(X, \Lambda)$. By dévissage we may also assume K is perverse. By [KrW, sect. 4.2] the Euler characteristic of K is equal, as an element of $End_{\mathbf{D}}(\mathbf{1}) = \Lambda$, to the categorical trace $tr(K) = tr(id_K)$ given by the rigid monoidal structure of $(\mathbf{D}, *)$ as in section 5 below. Categorical traces are preserved by tensor equivalences of rigid symmetric monoidal categories, so by proposition 9 we are done. \square

5. THE ANDRÉ-KAHN QUOTIENT

In this section we assume that the category \mathbf{D} and the functor rat satisfy axioms (1)-(5) of section 3. We then obtain a semisimple abelian rigid symmetric monoidal category $\bar{\mathbf{D}}$ using a general construction of [AK1]. Let us briefly recall this construction in the present context.

By rigidity, any endomorphism f of an object K in \mathbf{D} has an adjoint morphism $f^{\sharp} : \mathbf{1} \rightarrow K * K^{\vee}$. The trace $tr(f) \in End_{\mathbf{D}}(\mathbf{1}) = \Lambda$ is defined as the composite $tr(f) = ev_K \circ S_{K, K^{\vee}} \circ f^{\sharp}$ where $S_{K, K^{\vee}} : K * K^{\vee} \rightarrow K^{\vee} * K$ denotes the symmetry constraint and where $ev_K : K^{\vee} * K \rightarrow \mathbf{1}$ is the evaluation. As in section 7.1 of loc. cit. we consider the André-Kahn radical $N_{\mathbf{D}}$ of \mathbf{D} , i.e. the ideal which is defined on objects K, L of \mathbf{D} by

$$N_{\mathbf{D}}(K, L) = \{f \in Hom_{\mathbf{D}}(K, L) \mid \forall g \in Hom_{\mathbf{D}}(L, K) : tr(g \circ f) = 0\}.$$

By definition, the quotient category

$$\bar{\mathbf{D}} = \mathbf{D}/N_{\mathbf{D}}$$

has the same objects K, L as \mathbf{D} , but $Hom_{\bar{\mathbf{D}}}(K, L) = Hom_{\mathbf{D}}(K, L)/N_{\mathbf{D}}(K, L)$. We denote by $\bar{\mathbf{P}}$ the essential image of \mathbf{P} in $\bar{\mathbf{D}}$.

Lemma 11. *The quotient functor $q : \mathbf{D} \rightarrow \bar{\mathbf{D}}$ preserves direct sums, and the category $\bar{\mathbf{P}}$ is pseudo-abelian (i.e. every idempotent in $\bar{\mathbf{P}}$ splits).*

Proof. The functor q preserves direct sums since it is Λ -linear. To see that idempotents in $\bar{\mathbf{P}}$ split, let P be an object of \mathbf{P} . Since \mathbf{P} is an abelian category, it suffices to show that every idempotent in

$$\text{End}_{\bar{\mathbf{P}}}(P) = \text{End}_{\mathbf{P}}(P)/N_{\mathbf{D}}(P, P)$$

lifts to an idempotent in $\text{End}_{\mathbf{P}}(P)$. Since \mathbf{P} is semisimple by axiom (5), we can assume $P = Q^{\oplus r}$ for some simple object Q of \mathbf{P} and $r \in \mathbb{N}$. Then $\text{End}_{\mathbf{P}}(P)$ is the ring of $r \times r$ matrices over the skew field $\text{End}_{\mathbf{P}}(Q)$. Since matrix rings over skew fields do not have proper two-sided ideals, it follows that either $N_{\mathbf{D}}(P, P) = 0$ or $N_{\mathbf{D}}(P, P) = \text{End}_{\mathbf{P}}(P)$. In both cases the lifting of idempotents is obvious. \square

Proposition 12. *The quotient category $\bar{\mathbf{D}}$ is a semisimple abelian Λ -linear rigid symmetric monoidal category.*

Proof. By lemma 7.1.1 in loc. cit. $N_{\mathbf{D}}$ is a monoidal ideal, so by sorite 6.1.4 of loc. cit. the quotient category $\bar{\mathbf{D}}$ is again a Λ -linear rigid symmetric monoidal category with $\text{End}_{\bar{\mathbf{D}}}(\mathbf{1}) = \Lambda$. We claim that

$$(5.1) \quad \text{Hom}_{\bar{\mathbf{D}}}(P[m], Q[n]) = 0 \quad \text{for all } P, Q \text{ in } \mathbf{P} \text{ and } m \neq n.$$

Indeed, for $m > n$ we even have $\text{Hom}_{\mathbf{D}}(P[m], Q[n]) = 0$ since under the faithful functor rat this Hom -group injects into

$$\text{Hom}_{D_c^b(X, \Lambda)}(\text{rat}(P)[m], \text{rat}(Q)[n]) = \text{Ext}_{\text{Perv}(X, \Lambda)}^{n-m}(\text{rat}(P), \text{rat}(Q))$$

which vanishes for $m > n$ (for the above interpretation as an Ext -group recall that $D_c^b(X, \Lambda)$ is the derived category of $\text{Perv}(X, \Lambda)$). For $m < n$ similarly $\text{Hom}_{\mathbf{D}}(Q[n], P[m]) = 0$, and in that case the definition of $N_{\mathbf{D}}$ trivially implies that $\text{Hom}_{\mathbf{D}}(P[m], Q[n]) = N_{\mathbf{D}}(P[m], Q[n])$. This is mapped to zero under the quotient functor $\mathbf{D} \rightarrow \bar{\mathbf{D}}$, hence (5.1) follows.

Now by axiom (4) every object K of $\bar{\mathbf{D}}$ can be written as $K = \bigoplus_{n \in \mathbb{Z}} K_n[n]$ with certain K_n in $\bar{\mathbf{P}}$. For such K the vanishing in (5.1) implies

$$(5.2) \quad \text{End}_{\bar{\mathbf{D}}}(K) = \bigoplus_{n \in \mathbb{Z}} \text{End}_{\bar{\mathbf{D}}}(K_n[n]) = \bigoplus_{n \in \mathbb{Z}} \text{End}_{\bar{\mathbf{P}}}(K_n).$$

In particular, every idempotent endomorphism of K in the category $\bar{\mathbf{D}}$ is a direct sum of idempotent endomorphisms of the summands $K_n[n]$, and by lemma 11 it follows that $\bar{\mathbf{D}}$ is pseudo-abelian. Hence to show that $\bar{\mathbf{D}}$ is a semisimple abelian category, it will suffice by [AK1, A.2.10] to show that it is a semisimple Λ -linear category in the sense of section 2.1.1 in loc. cit. For this we use the following general result [AK2, th. 1]:

Let F be a field and \mathbf{A} an F -linear rigid symmetric monoidal category with $\text{End}_{\mathbf{A}}(\mathbf{1}) = F$. Suppose there exists an F -linear tensor functor ACU from \mathbf{A} to an abelian F -linear rigid symmetric monoidal category \mathbf{V} such

that $\dim_{\Lambda}(\text{Hom}_{\mathbf{V}}(V_1, V_2)) < \infty$ for all $V_1, V_2 \in \mathbf{V}$. Then the quotient of \mathbf{A} by its André-Kahn radical $N_{\mathbf{A}}$ is a semisimple F -linear category, and $N_{\mathbf{A}}$ is the unique monoidal ideal of \mathbf{A} with this property.

In our case this applies for $F = \Lambda$, $\mathbf{A} = \mathbf{D}$ and for the functor $H^{\bullet}(X, -)$ from \mathbf{D} to the abelian category \mathbf{V} of super vector spaces over Λ . \square

Corollary 13. *The functors $\mathbf{P} \rightarrow \bar{\mathbf{P}}$ and $\bar{\mathbf{P}} \hookrightarrow \bar{\mathbf{D}}$ are exact functors between semisimple abelian categories. Simple objects of \mathbf{P} either remain simple or become isomorphic to zero in $\bar{\mathbf{P}}$.*

Proof. By proposition 12, $\bar{\mathbf{D}}$ is a semisimple abelian category, and it also follows from the proof of the proposition that $\bar{\mathbf{P}}$ is a semisimple abelian subcategory of $\bar{\mathbf{D}}$. Since the considered functors are additive, they are exact by semisimplicity. If P is a simple object of \mathbf{P} , then $\text{End}_{\mathbf{P}}(P)$ is a skew field, hence $\text{End}_{\bar{\mathbf{P}}}(P)$ is a skew field or zero, and P is simple or zero in $\bar{\mathbf{P}}$. \square

Remark 14. *The endofunctors $K \mapsto K_{\chi}$ of \mathbf{D} induce tensor equivalences of the symmetric monoidal category $\bar{\mathbf{D}}$.*

6. SUPER-TANNAKIAN CATEGORIES

Suppose that Λ is algebraically closed and that axioms (1)-(5) of section 3 hold. As in section 5 we then have the abelian Λ -linear rigid symmetric monoidal category $\bar{\mathbf{D}}$. By semisimplicity the functor $*$: $\bar{\mathbf{D}} \times \bar{\mathbf{D}} \rightarrow \bar{\mathbf{D}}$ is exact in each variable, and $\text{End}_{\bar{\mathbf{D}}}(\mathbf{1}) = \Lambda$ (this latter property is inherited from \mathbf{D} where it can be checked by applying *rat*). So $\bar{\mathbf{D}}$ is a *catégorie Λ -tensorielle* in the sense of [De2, sect. 0.1].

Recall that a full subcategory of $\bar{\mathbf{D}}$ is said to be finitely tensor generated if it is the category of all subquotients of convolution powers of $C \oplus C^{\vee}$ for some fixed object C . We claim that any such category is super-Tannakian in the following sense.

The framework of algebraic geometry is generalized to super algebraic geometry by replacing the category of rings with the one of $\mathbb{Z}/2\mathbb{Z}$ -graded supercommutative rings. In particular one has the notions of algebraic and reductive supergroups over Λ and their super representations, as we recall in the appendix in section 14 below. Given an algebraic supergroup G over Λ and $\varepsilon \in G(\Lambda)$ with $\varepsilon^2 = 1$ such that $\text{int}(\varepsilon)$ is the parity automorphism of G , we denote by $\text{Rep}_{\Lambda}(G, \varepsilon)$ the category of super representations $V = V_+ \oplus V_-$ of G over Λ for which ε acts by ± 1 on V_{\pm} . Such categories will be called super-Tannakian, with Tannaka supergroup G .

Theorem 15. *Every finitely generated full tensor subcategory \mathbf{T} of $\bar{\mathbf{D}}$ is super-Tannakian, with a reductive Tannaka supergroup $G = G(\mathbf{T})$.*

Proof. Since $\bar{\mathbf{D}}$ is a catégorie Λ -tensorielle, for the first claim it suffices by [De2, th. 0.6] to see that for any object $C \in \bar{\mathbf{D}}$ the number of constituents of C^{*n} is at most N^n for some constant $N = N(C)$ and all $n \in \mathbb{N}$. For this one

can take $N(C) = \sum_{i \in \mathbb{Z}} \dim_{\Lambda}(H^i(X, D))$ with any object $D \in \mathbf{D}$ that becomes isomorphic to C in $\bar{\mathbf{D}}$, see [We2, top of p. 5]. Concerning reductivity, note that by [We3] a category $\text{Rep}_{\Lambda}(G, \varepsilon)$ is semisimple iff G is reductive. \square

Corollary 16. *Let $\mathbf{N} \subseteq \mathbf{D}$ be the full subcategory of all objects that become isomorphic to zero in $\bar{\mathbf{D}}$. Then \mathbf{N} satisfies axioms (N1), (N3) and (N4), and an object $K \in \mathbf{D}$ is in \mathbf{N} iff all simple perverse constituents of all perverse cohomology sheaves ${}^p H^n(K)$ have Euler characteristic zero.*

Proof. Property (N1) is obvious. Property (N3) follows from (N4). To show (N4) let P be a simple object of \mathbf{P} with Euler characteristic $\chi(P) = 0$. It lies in some super-Tannakian subcategory of $\bar{\mathbf{D}}$ as in theorem 15. The super representation corresponding to P is irreducible by corollary 13, and by [KrW, 4.2] its superdimension is $\chi(P) = 0$. So (N4) follows because by [We3, lemma 15, p. 29] reductive supergroups do not admit non-zero irreducible super representations of superdimension zero. Conversely, in view of (N1) it remains to show $\chi(P) = 0$ for any simple object P of \mathbf{P} contained in \mathbf{N} . This follows from $\chi(P) = \text{tr}(\text{id}_P)$. Indeed $\text{id}_P \in N_{\mathbf{D}}(P, P)$ since $P \in \mathbf{N}$. Hence $\text{tr}(\text{id}_P \circ \text{id}_P) = 0$, and therefore $\chi(P) = 0$. \square

Corollary 17. *If the base field k is \mathbb{C} and if also axiom (6) holds, then the supergroup $G = G(\mathbf{T})$ in theorem 15 is a classical reductive algebraic group, and $G(\mathbf{T}_{\chi}) \cong G(\mathbf{T})$ for twists by arbitrary characters χ of $\pi_1(X, 0)$.*

Proof. Corollary 16 and theorem 20 show that the category $\bar{\mathbf{P}}$ is preserved under convolution. For $k = \mathbb{C}$ and $\mathbf{T} \subset \bar{\mathbf{P}}$ the assertion therefore follows from [De1, th. 7.1] and remark 14, since $\chi(K) \geq 0$ for all $K \in \bar{\mathbf{P}}$ if $k = \mathbb{C}$ (see section 8). The general case can be reduced to the special case $\mathbf{T} \subset \bar{\mathbf{P}}$. \square

At present we do not know whether the corresponding statement also holds for $\text{char}(k) > 0$. The problem is that in positive characteristic we do not know how to prove proposition 21 below for simple abelian varieties.

7. PERVERSE MULTIPLIER

In this section Λ need not be algebraically closed, but we assume that axioms (1)-(5) of section 3 hold. We also assume that we are given a full subcategory $\mathbf{N} \subseteq \mathbf{D}$ satisfying (N1). We define an \mathbf{N} -multiplier to be an object $K \in \mathbf{D}$ such that for all $r \in \mathbb{N}_0$ and all $n \neq 0$ the subquotients of ${}^p H^n((K \oplus K^{\vee})^{*r})$ are all contained in \mathbf{N} . The relevance of this notion becomes clear from the following lemma, where $K \in \mathbf{D}$ is called a zero type if $H^n(X, K) = 0$ holds for $n \neq 0$.

Lemma 18. *Let $P \in \mathbf{P}$.*

- a) *If \mathbf{N} satisfies (N2) and if P is an \mathbf{N} -multiplier, then P_{χ} is a zero type for most χ .*
- b) *Conversely, assuming axiom (6), if P_{χ} is a zero type for some χ and if the twisted category \mathbf{N}_{χ} satisfies (N3), then P is an \mathbf{N} -multiplier.*

Proof. a) Put $g = \dim(X)$ and $P^{*(g+1)} = \bigoplus_{m \in \mathbb{Z}} P_m[m]$ with P_m in \mathbf{P} . Since P is an \mathbf{N} -multiplier, we have $P_m \in \mathbf{N}$ for $m \neq 0$. By (N2) it follows that for most χ and all $n \in \mathbb{Z}$,

$$H^n(X, P_\chi^{*(g+1)}) = H^n(X, (P_0)_\chi).$$

The right hand side vanishes for $|n| > g$, since $\text{rat}((P_0)_\chi)$ is a perverse sheaf. But for the left hand side we have

$$H^\bullet(X, P_\chi^{*(g+1)}) = (H^\bullet(X, P_\chi))^{\otimes(g+1)}$$

by proposition 9 and since $H^\bullet(X, -)$ is a tensor functor [We1, p. 27]. So the above vanishing statement for $|n| > g$ implies that P_χ is a zero type.

b) Clearly P is an \mathbf{N} -multiplier iff P_χ is an \mathbf{N}_χ -multiplier, so in view of proposition 9 we can assume χ to be the trivial character. Then P is a zero type, and for any $r \in \mathbb{N}_0$ the tensor functoriality of $H^\bullet(X, -)$ implies that $Q = (P \oplus P^\vee)^{*r}$ is a zero type. Using the hard Lefschetz axiom (6), one deduces that for $n \neq 0$ the hypercohomology $H^\bullet(X, {}^p H^n(Q))$ of ${}^p H^n(Q)$ vanishes. So P is an \mathbf{N} -multiplier by (N3). \square

The following consequence of the hard Lefschetz axiom (6) will be a crucial observation for our proof of theorem 2.

Lemma 19. *If \mathbf{N} satisfies (N1) and (N4) and \mathbf{D} satisfies (1)-(6) and if $P \in \mathbf{P}$ is not an \mathbf{N} -multiplier, then for some $r \in \mathbb{N}$ the convolution $(P * P^\vee)^{*r}$ admits a direct summand $\mathbf{1}[2i](i)$ with $i \neq 0$.*

Proof. If P is not an \mathbf{N} -multiplier, there are $a, b \in \mathbb{N}$ such that $P^{*a} * (P^\vee)^{*b}$ admits a direct summand $Q[i]$, for some $i \neq 0$ and some simple object $Q \in \mathbf{P}$ which is not in \mathbf{N} . By the hard Lefschetz axiom (6) then $Q-i$ is a direct summand of $P^{*a} * (P^\vee)^{*b}$. So the dual Q^\veei is a direct summand of $P^{*b} * (P^\vee)^{*a}$. Altogether then $Q[i] * Q^\vee[i](-i) = Q * Q^\vee[2i](i)$ is a direct summand of $(P * P^\vee)^{*r}$ for $r = a + b$.

It remains to show that $\mathbf{1}$ is a direct summand of $Q * Q^\vee$. For this note that the trace map $\text{tr}(Q) : \mathbf{1} \rightarrow Q * Q^\vee \cong Q^\vee * Q \rightarrow \mathbf{1}$ is non-zero, since $\chi(Q) \neq 0$ by axiom (N4). Note that $\text{tr}(Q)$ factors over ${}^p H^0(Q * Q^\vee)$; indeed, using the faithful functor rat , one has $\text{Hom}_{\mathbf{D}}(\mathbf{P}, {}^p \tau_{>0} \mathbf{P}) = \text{Hom}_{\mathbf{D}}({}^p \tau_{<0} \mathbf{P}, \mathbf{P}) = 0$. So $\text{tr}(Q)$ exhibits $\mathbf{1}$ as a retract of ${}^p H^0(Q * Q^\vee)$ in the abelian category \mathbf{P} . \square

8. PROOF OF THEOREM 2

To prove theorem 2, by dévissage we can restrict ourselves to semisimple perverse sheaves as in example 6. So suppose that $\Lambda = \mathbb{C}$ or $\Lambda = \overline{\mathbb{Q}}_l$ and that \mathbf{D} satisfies the axioms (1)-(6) of section 3. Consider the semisimple abelian rigid symmetric monoidal quotient category $\overline{\mathbf{D}}$ as in section 5.

For the full subcategory $\mathbf{N} \subseteq \mathbf{D}$ of all objects that become isomorphic to zero in $\overline{\mathbf{D}}$, the axioms (N1), (N3) and (N4) hold by corollary 16. If the base field k is \mathbb{C} , then also axiom (N2) holds by corollary 22 below. So theorem 2 is an immediate consequence of lemma 18a) together with the following

Theorem 20. *Let $\mathbf{N} \subseteq \mathbf{D}$ be a full subcategory satisfying the axioms (N1) and (N4). Then every object $P \in \mathbf{P}$ is an \mathbf{N} -multiplier.*

Proof. Suppose that $P \in \mathbf{P}$ is simple and not an \mathbf{N} -multiplier. Then for some integer $r \in \mathbb{N}$ the convolution $(P * P^\vee)^{*r}$ contains by lemma 19 a direct summand $L = \mathbf{1}[2i](i)$ with $i \neq 0$. So the full rigid symmetric monoidal subcategory $\bar{\mathbf{D}}_1$ of $\bar{\mathbf{D}}$ generated by P contains the full rigid symmetric monoidal subcategory $\bar{\mathbf{D}}_0$ generated by the invertible object L .

Theorem 15 shows that for certain reductive supergroups G_i over Λ we have tensor equivalences $\omega_i : \bar{\mathbf{D}}_i \xrightarrow{\sim} \text{Rep}_\Lambda(G_i, \varepsilon_i)$ for $i \in \{0, 1\}$, and by the Tannakian formalism the inclusion $\bar{\mathbf{D}}_0 \subseteq \bar{\mathbf{D}}_1$ defines an epimorphism of reductive supergroups

$$h : G_1 \twoheadrightarrow G_0.$$

The category $\bar{\mathbf{D}}_0$ consists of all direct sums of skyscraper sheaves L^{*n} with integers $n \in \mathbb{Z}$. Since $L^{*n} \cong \mathbf{1}[2ni](ni)$ and $i \neq 0$, equation (5.2) implies that one has $L^{*n} \cong \mathbf{1}$ in $\bar{\mathbf{D}}$ only if $n = 0$. Taking into account that the symmetry constraint $S_{L,L} : L * L \rightarrow L * L$ is the identity in \mathbf{D} , the tensor equivalence ω_0 between $\bar{\mathbf{D}}_0$ and $\text{Rep}_\Lambda(\mathbb{G}_m, 1)$ is realized explicitly, with the multiplicative Tannaka group $G_0 = \mathbb{G}_m$ and $\varepsilon_0 = 1$, via

$$L^{*n} \mapsto (\text{the character } z \mapsto z^n \text{ of } \mathbb{G}_m).$$

In particular, the representation $W_0 = \omega_0(L)$ is non-trivial.

But proposition 38 in the appendix applies to the torus $T_0 = G_0 = \mathbb{G}_m$, so there exists a central torus $T_1 \cong \mathbb{G}_m$ in G_1 such that $h : G_1 \rightarrow G_0$ restricts to an isogeny $T_1 \rightarrow T_0$. By Schur's lemma the central torus T_1 acts via some character on the irreducible super representation $W_1 = \omega_1(P)$; so T_1 acts trivially on $W_1 \otimes W_1^\vee = \omega_1(P * P^\vee)$. Then T_1 , hence also T_0 , acts trivially also on the direct summand $W_0 \subseteq (W_1 \otimes W_1^\vee)^{\otimes r}$ – a contradiction. \square

9. EULER CHARACTERISTICS

In this section we assume $k = \mathbb{C}$. By [FK, cor. 1.4] then every perverse sheaf P on X has Euler characteristic $\chi(P) \geq 0$. Generalizing the argument of loc. cit., we determine all P such that $\chi(P) \in \{0, 1\}$.

Proposition 21. *Let P be a simple perverse sheaf on X .*

- a) *One has $\chi(P) = 0$ iff there exists an isogeny $f : X_1 \times X_2 \rightarrow X$ with $d = \dim(X_1) > 0$ such that*

$$f^*(P) \cong L_\varphi[d] \boxtimes Q,$$

for the local system L_φ on X_1 attached to some character φ of the fundamental group $\pi_1(X_1, 0)$ and for some $Q \in \text{Perv}(X_2, \mathbb{C})$.

- b) *One has $\chi(P) = 1$ iff P is a skyscraper sheaf on X of rank one.*

Proof. View P as a D_X -module via the Riemann-Hilbert correspondence. For $Z \subseteq X$ closed and irreducible, let $\Lambda_Z \subseteq T^*X$ be the closure in T^*X of the conormal bundle in X to the smooth locus of Z . As in loc. cit. we write the characteristic cycle of P as a finite formal sum

$$CC(P) = \sum_{Z \subseteq X} n_Z \cdot \Lambda_Z \quad \text{with} \quad n_Z \in \mathbb{N}_0,$$

where Z runs through all closed irreducible subsets of X . From $CC(P)$ the support of the perverse sheaf P can be recovered via $\text{Supp } P = \bigcup_{n_Z \neq 0} Z$. Furthermore, by the microlocal index formula [Gi, th. 9.1],

$$\chi(P) = \sum_{Z \subseteq X} n_Z \cdot d_Z \quad \text{with} \quad d_Z = [\Lambda_X] \cdot [\Lambda_Z] \in \mathbb{Z}.$$

The intersection numbers d_Z are well-defined even though Λ_Z is not proper for $Z \neq X$; see loc. cit. for details. Now part *b*) follows from lemma 23 below. If X is a simple abelian variety, then part *a*) follows from the same lemma combined with lemma 24. The non-simple case can be reduced to the simple case, see [We4]. \square

The reduction step to the case of simple abelian varieties in [We4] works for ground fields k of characteristic $p > 0$ as well, but we do not know how to deal with simple abelian varieties in that case. For $k = \mathbb{C}$, Christian Schnell has given in [Schn, cor. 3.11] a different proof of proposition 21a) via the theory of \mathcal{D} -modules.

Corollary 22. *The Euler characteristic of a simple perverse sheaf P on X vanishes iff $H^\bullet(X, P_\chi) = 0$ holds for most characters χ .*

Proof. “ \Leftarrow ” follows from corollary 10. For “ \Rightarrow ” we use proposition 21a). For characters $\chi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$ put $\chi \circ f_* = \chi_1 \boxtimes \chi_2$ for characters χ_i of $\pi_1(X_i, 0)$. Then P_χ is a direct summand of $Rf_*(f^*(P_\chi))$, hence a direct summand of

$$Rf_*(L_\varphi[d] \boxtimes Q)_\chi = Rf_*(L_\varphi[d](\chi_1) \boxtimes Q(\chi_2))$$

with φ and Q as in the proposition. By the Künneth formula therefore $H^\bullet(X, P_\varphi)$ vanishes if

$$H^\bullet(Y, L_\varphi[d](\chi_1)) \otimes H^\bullet(X_2, Q(\chi_2)) = 0.$$

But indeed $H^\bullet(Y, L_\varphi[d](\chi_1)) = H^\bullet(Y, L_{\varphi \cdot \chi_1}[d]) = 0$ for $\chi_1 \neq \varphi^{-1}$. \square

Lemma 23. *With notations as above, $d_Z \geq 0$ holds for all Z . One has $d_Z = 1$ iff Z is reduced to a single point. If X is simple, then $d_Z = 0$ iff $Z = X$.*

Proof. The cotangent bundle $T^*X = X \times \mathbb{C}^g$ is trivial of rank $g = \dim(X)$, and projecting from $\Lambda_Z \subseteq T^*X$ onto the second factor \mathbb{C}^g induces the Gauß mapping $p : \Lambda_Z \rightarrow \mathbb{C}^g$. By [FK, prop. 2.2] the intersection number d_Z is the generic degree of p . In particular $d_Z \geq 0$.

If $d_Z = 1$, then Λ_Z is birational to \mathbb{C}^g , so by [Mi1, cor. 3.9] there does not exist any non-constant map from Λ_Z to an abelian variety. So the image Z of the composite map $\Lambda_Z \subseteq T^*X \rightarrow X$ is a single point.

If $d_Z = 0$, then p is not surjective, so $\dim(p(\Lambda_Z)) < g$. Then for some cotangential vector $\omega \in p(\Lambda_Z)$ the fibre $p^{-1}(\omega)$ is positive-dimensional. If $Z \neq X$, we can assume $\omega \neq 0$. Let $Y \subseteq X$ be the image of $p^{-1}(\omega) \subseteq T^*X$ under the map $T^*X \rightarrow X$. Then $\dim(Y) > 0$, and up to a translation we can assume $0 \in Y$. By construction ω is normal to Y in every smooth point of Y , so the preimage of Y under the universal covering $\mathbb{C}^g \rightarrow X = \mathbb{C}^g/\Lambda$ lies in the hyperplane of \mathbb{C}^g orthogonal to ω . Thus the abelian subvariety of X generated by Y is strictly contained in X , but non-zero, contradicting the assumption that X is simple. \square

Lemma 24. *Let P be a simple perverse sheaf on X . If there exists a closed subset $Y \subset X$ with $\dim(Y) \leq g - 2$ such that*

$$CC(P) = n_X \Lambda_X + \sum_{Z \subseteq Y} n_Z \Lambda_Z \quad \text{and} \quad n_X > 0,$$

then $P = L_\chi[g]$ for the local system L_χ on X attached to some character χ of $\pi_1(X, 0)$. In particular then $n_X = 1$ and $n_Z = 0$ for $Z \neq X$.

Proof. Consider $j : U = X \setminus Y \hookrightarrow X$. Since open embeddings are non-characteristic for any D_X -module, theorem 2.4.6 and remark 2.4.8 in [HTT] show $CC(j^*(P)) = CC(P) \cap T^*U = n_X \cdot \Lambda_U$. By prop. 2.2.5 in loc. cit. then $j^*(P) = L_U[g]$ for some local system L_U on U . Since X is smooth, by the purity of branch points the assumption on $\dim(Y)$ implies $L_U = j^*(L)$ for some local system L on X . By simplicity of $P = j_{!*}(j^*(P))$ then L has rank one and $P = L[g]$. \square

10. LOCALIZATION AT HEREDITARY CLASSES

In this section k and Λ can be arbitrary, but we assume that \mathbf{D} and the functor rat satisfy the condition:

(T) *Triangulation.* The category \mathbf{D} is triangulated and has a t -structure with core \mathbf{P} which induces properties (1) and (2).

We say a class \mathbf{H} of simple objects in \mathbf{P} is *hereditary* if for $K \in \mathbf{H}$ and $L \in \mathbf{D}$ all simple constituents of ${}^pH^n(K * L)$ are in \mathbf{H} for all $n \in \mathbb{Z}$, and if $K \in \mathbf{H}$ implies $K^\vee \in \mathbf{H}$. By dévissage, it suffices to know the first condition for all simple objects $L \in \mathbf{P}$. A union of hereditary classes is hereditary, and character twists H_χ of hereditary classes \mathbf{H} are hereditary.

For example, axiom (S) in section 11 below implies that the class H_{coh} of simple objects K with $H^\bullet(X, K) = 0$ is hereditary: For simple L in \mathbf{P} and K in H_{coh} we have $H^\bullet(X, K * L) = H^\bullet(X, K) \otimes H^\bullet(X, L) = 0$ by the Künneth formula, so by semisimplicity all simple constituents of ${}^pH^n(K * L)$ lie again in H_{coh} . Similarly, if (S) and axiom (3) of section 3 hold, the class H_{most} of simple objects K for which most character twists K_χ are in H_{coh} is

hereditary. The class H_{Euler} of simple objects with $\chi(K) = 0$ is hereditary in the situation of corollary 16. For a fixed abelian subvariety $A \subseteq X$, the class H_A of all A -equivariant simple objects is hereditary [KW, III.15.6]. Note that $H_{most} \subseteq H_{Euler}$ and that for $k = \mathbb{C}$ we have $H_{coh}, H_A \subseteq H_{most} = H_{Euler}$ by corollary 22.

If H is hereditary, let $\mathbf{N}_H \subseteq \mathbf{D}$ be the full subcategory of all $K \in \mathbf{D}$ such that all simple constituents of all ${}^pH^n(K)$ are isomorphic to objects in H . Then \mathbf{N}_H is a thick triangulated tensor ideal of $(\mathbf{D}, *)$, so the localization $\mathbf{D} \rightarrow \mathbf{D}_H = \mathbf{D}[\Sigma^{-1}]$, with respect to the class Σ of all morphisms with cones in \mathbf{N}_H , inherits the structure of a rigid symmetric monoidal category and this localization is a tensor functor. $\mathbf{P} \cap \mathbf{N}$ is a Serre subcategory of the abelian category \mathbf{P} , i.e. it is closed under extensions and subquotients.

The images of ${}^p\tau_{\leq 0}(\mathbf{D})$ and ${}^p\tau_{\geq 1}(\mathbf{D})$ under the functor $\mathbf{D} \rightarrow \mathbf{D}_H$ define a t -structure on \mathbf{D}_H . Let us check the crucial property $Hom_{\mathbf{D}_H}(K, M) = 0$ for objects $K \in {}^p\tau_{\leq 0}(\mathbf{D})$ and $M \in {}^p\tau_{\geq 1}(\mathbf{D})$: Morphisms in $Hom_{\mathbf{D}_H}(K, M)$ are represented by diagrams

$$(10.1) \quad fs^{-1} : K \xleftarrow{s} L \xrightarrow{f} M$$

for $s \in \Sigma$ and $f \in Hom_{\mathbf{D}}(L, M)$. Now $Cone(s) \in \mathbf{N}_H$ and $K \in {}^p\tau_{\leq 0}(\mathbf{D})$, hence ${}^pH^n(L) \in \mathbf{N}_H$ for all $n \geq 1$. So the truncation morphism $s' : {}^p\tau_{\leq 0}(L) \rightarrow L$ lies in Σ , and $fs^{-1} = (fs')(ss')^{-1}$ by the calculus of fractions. Replacing L by ${}^p\tau_{\leq 0}(L)$ we may thus assume $L \in {}^p\tau_{\leq 0}(\mathbf{D})$. Then $f \in Hom_{\mathbf{D}}(L, M) = 0$ and hence $fs^{-1} = 0$ in \mathbf{D}_H since we have $M \in {}^p\tau_{\geq 1}(\mathbf{D})$. This being said, it is clear that the core of the t -structure on \mathbf{D}_H is the image \mathbf{P}_H of \mathbf{P} . For $H = H_*$ with $\star \in \{Euler, most, coh, A\}$ we put $\mathbf{N}_\star = \mathbf{N}_H$, $\mathbf{D}_\star = \mathbf{D}_H$ and $\mathbf{P}_\star = \mathbf{P}_H$.

Lemma 25. *The morphisms in \mathbf{P}_H are obtained by inverting the morphisms s in \mathbf{P} whose kernel and cokernel are in $\mathbf{P} \cap \mathbf{N}$. Hence $\mathbf{P} \rightarrow \mathbf{P}_H$ is the abelian category quotient $\mathbf{P} \rightarrow \mathbf{P}/(\mathbf{P} \cap \mathbf{N})$ defined in [Ga, p. 364ff.].*

Proof. For morphisms $fs^{-1} \in Hom_{\mathbf{D}_H}(K, M)$ between objects $K, M \in \mathbf{P}$, represented by a diagram (10.1), we may assume that $L \in {}^p\tau_{\leq 0}(\mathbf{D})$. The adjunction morphism $t : L \rightarrow {}^p\tau_{\geq 0}(L) = {}^pH^0(L)$ is in Σ and $f = f' \circ t$ for some $f' : {}^pH^0(L) \rightarrow M$ since $M \in \mathbf{P} \subset {}^p\tau_{\geq 0}(\mathbf{D})$. We can replace L by ${}^pH^0(L)$ using the calculus of fractions. Then $L \in \mathbf{P}$, so the cokernel N of the morphism $i : K' = im(s) \rightarrow K$ is in \mathbf{N}_H and also $N' = f(ker(s)) \in \mathbf{N}_H$. The diagram

$$\begin{array}{ccccccc} L & \dashrightarrow & K' & \xrightarrow{i} & K & \twoheadrightarrow & N \\ & & & & \searrow k & & \\ & & N' & \hookrightarrow & M & \xrightarrow{j} & M' \end{array}$$

defines a morphism in $Hom_{\mathbf{P}/(\mathbf{P} \cap \mathbf{N}_H)}(K, M)$, i.e. an equivalence class of triples (i, j, k) for a monomorphism $i : K' \rightarrow K$, an epimorphism $j : M \rightarrow M'$ and a morphism $k : K' \rightarrow M'$ with $i, j \in \Sigma$, see [Ga, p. 364ff.]. It is easy

to see that the assignment $(fs^{-1})/\sim \mapsto (i, j, k)/\sim$ induces an equivalence between \mathbf{P}_H and $\mathbf{P}/(\mathbf{P} \cap \mathbf{N}_H)$. \square

11. THE TANNAKA GROUPS $G(K)$ AND $G(X)$

Let Λ be algebraically closed. We then generalize the construction of section 6 to the non-semisimple case, assuming axiom (T) of section 10, axiom (3) of section 3 and the following property:

(S) *Semisimple objects.* The full subcategory $\mathbf{D}^{ss} \subseteq \mathbf{D}$ of finite direct sums of degree shifts of simple objects in \mathbf{P} satisfies axioms (1)-(6).

For example, these assumptions are valid if $\mathbf{D} = D_c^b(X, \Lambda)$. Let $\mathbf{P}^0 \subseteq \mathbf{P}$ be the full subcategory of all objects all of whose simple constituents are zero types, and denote by \mathbf{P}_{coh}^0 the essential image of this subcategory in \mathbf{P}_{coh} .

Lemma 26. \mathbf{P}_{coh}^0 is a rigid symmetric monoidal abelian subcategory of the rigid symmetric monoidal category \mathbf{D}_{coh} .

Proof. Clearly \mathbf{P}_{coh}^0 is an abelian subcategory of \mathbf{P}_{coh} . Since \mathbf{D}_{coh} is a rigid symmetric monoidal category, it remains to show $\mathbf{P}_{coh}^0 * \mathbf{P}_{coh}^0 \subseteq \mathbf{P}_{coh}^0$. For this it will by dévissage suffice to see that the convolution of *simple* objects of \mathbf{P}^0 lies again in \mathbf{P}^0 . But this is clear, since such a convolution is again a zero type and since it is semisimple by axiom (S). \square

Given finitely many objects $K_1, \dots, K_r \in \mathbf{P}$, consider $K = \bigoplus_{i=1}^r K_i \oplus K_i^\vee$. Assume that $K_\chi \in \mathbf{P}^0$ for some character χ . By theorem 2 this holds for most χ if $k = \Lambda = \mathbb{C}$. Let

$$\langle K_\chi \rangle \subseteq \mathbf{P}_{coh}^0$$

be the rigid symmetric monoidal abelian subcategory generated in \mathbf{P}_{coh}^0 by the image of K under the functor $\mathbf{P}^0 \rightarrow \mathbf{P}_{coh}^0$. Then by [De1, th. 7.1] the category $\langle K_\chi \rangle$ is Tannakian, since $tr(id_M) = \dim_\Lambda(H^0(X, M)) \geq 0$ holds for all M in \mathbf{P}_{coh}^0 . So, for some affine algebraic group $G(K)$ over Λ ,

$$\langle K_\chi \rangle = Rep_\Lambda(G(K)).$$

Since Λ is algebraically closed, we have the following

Lemma 27. Up to isomorphism, the algebraic group $G(K)$ depends only on the object K but not on the character χ .

Proof. For another character φ with $K_\varphi \in \mathbf{P}^0$, twisting by $\rho = \varphi/\chi$ is a tensor auto-equivalence of \mathbf{D} by proposition 9. Although it does not descend to \mathbf{D}_{coh} , we claim that it induces a tensor equivalence between $\langle K_\chi \rangle \subseteq \mathbf{P}_{coh}^0$ and $\langle K_\varphi \rangle \subseteq \mathbf{P}_{coh}^0$. Indeed, the objects of $\langle K_\chi \rangle$ are represented by the images in \mathbf{P}_{coh} of the subquotients of ${}^pH^0(K_\chi^{*r})$ with $r \in \mathbb{N}$. By lemma 25, morphisms in \mathbf{P}_{coh} between two such subquotients K and M are given by equivalence classes of diagrams in \mathbf{P} with exact rows and with

$N, N' \in \mathbf{N}_{coh} \subseteq \mathbf{N}_{Euler}$ as in the proof of lemma 25. Twisting by ρ gives a similar diagram in \mathbf{P}

$$\begin{array}{ccccc} K'_\rho & \hookrightarrow & K_\rho & \twoheadrightarrow & N_\rho \\ & \searrow & & & \\ N'_\rho & \hookrightarrow & M_\rho & \twoheadrightarrow & M'_\rho \end{array}$$

where $N_\rho, N'_\rho \in \mathbf{P}^0$ since both are subquotients of ${}^p H^0(K_\varphi^{*r})$ and $K_\varphi \in \mathbf{P}^0$ holds by our choice of φ . Twists preserve \mathbf{N}_{Euler} by corollary 10. Hence N_ρ, N'_ρ are in $\mathbf{N}_{Euler} \cap \mathbf{P}^0 = \mathbf{N}_{coh}$. So the twist of our diagram defines a morphism in \mathbf{P}_{coh} from K_ρ to M_ρ . This shows that twisting by ρ defines a tensor equivalence from $\langle K_\chi \rangle$ to $\langle K_\varphi \rangle$. Therefore replacing χ by φ only amounts to a change of fiber functor; since Λ is algebraically closed, this does not affect the Tannaka group. \square

Recall that throughout this section we always assume that $K_\chi \in \mathbf{P}^0$. The tensor functor $H^\bullet(X, \text{rat}(-))$ from \mathbf{D} to the category of super vector spaces over Λ factors over \mathbf{D}_{coh} . Since all objects of $\langle K_\chi \rangle$ are zero types, we thus get a tensor functor

$$H : \langle K_\chi \rangle \longrightarrow \text{Vec}_\Lambda$$

to the category Vec_Λ of finite-dimensional vector spaces over Λ . This is a fiber functor for $\langle K_\chi \rangle$. Indeed H is exact, since exact sequences in \mathbf{D}_{coh} are defined by distinguished triangles and objects of $\langle K_\chi \rangle$ are zero types. If $H(M) = 0$ for some $M \in \langle K_\chi \rangle$, then by exactness of H also $H(M_i) = 0$ for all constituents M_i of M , hence we have $M = 0$ in \mathbf{D}_{coh} . Therefore H is faithful. If $\text{rat}(K)$ is defined over a subfield $\Lambda_0 \subseteq \Lambda$, then H descends to a fiber functor over Λ_0 . This allows to define the Tannaka group of $\langle K_\chi \rangle$ as an algebraic group over Λ_0 , but then its isomorphism class may depend on χ .

Lemma 28. *For $K \in \mathbf{D}^{ss} \cap \mathbf{P}$ the Tannaka group $G(K)$ is isomorphic over Λ to the Tannaka group $G(\mathbf{T})$ of the symmetric monoidal subcategory \mathbf{T} , generated by K in the André-Kahn quotient of \mathbf{D}^{ss} in the sense of section 6.*

Proof. Let \mathbf{P}_{coh}^{ss} be the essential image of $\mathbf{P}^{ss} = \mathbf{P} \cap \mathbf{D}^{ss}$ in \mathbf{P}_{coh} , and denote by $\bar{\mathbf{P}}^{ss}$ the image of \mathbf{P}^{ss} in the André-Kahn quotient of \mathbf{D}^{ss} as defined in section 5. $\bar{\mathbf{P}}^{ss}$ is a semisimple abelian category, and it is also a symmetric monoidal category by theorem 20. By corollary 13 the functor $\mathbf{P}^{ss} \rightarrow \bar{\mathbf{P}}^{ss}$ is exact. So if s is a morphism in \mathbf{P}^{ss} whose kernel and cokernel lie in \mathbf{N}_{coh} , then the kernel and cokernel of the corresponding morphism in $\bar{\mathbf{P}}^{ss}$ are zero, i.e. the morphism s becomes invertible in $\bar{\mathbf{P}}^{ss}$. From the universal property of the localization \mathbf{P}_{coh}^{ss} we thus obtain a unique functor p such that the following diagram commutes, where q and \bar{q} denote the natural quotient

functors.

$$\begin{array}{ccc}
 & \mathbf{P}^{ss} & \\
 q \swarrow & & \searrow \bar{q} \\
 \mathbf{P}_{coh}^{ss} & \xrightarrow{\exists! p} & \bar{\mathbf{P}}^{ss}
 \end{array}$$

Now p is a tensor functor ACU, since q and \bar{q} are, and induces a functor p_K from the semisimple abelian subcategory $\langle K_\chi \rangle \subseteq \mathbf{P}_{coh}^{ss}$ to $\mathbf{T}_\chi \subseteq \bar{\mathbf{P}}^{ss}$ which is essentially surjective, since it is the identity on objects. One easily checks that it is fully faithful. Now conclude as in the proof of corollary 17. \square

For $K, L \in \mathbf{P}^0$ such that $\langle K \rangle \subseteq \langle L \rangle \subseteq \mathbf{P}_{coh}^0$, the Tannakian formalism gives an epimorphism $p_{LK} : G(L) \twoheadrightarrow G(K)$ of algebraic groups. Write \mathbf{P}_{coh}^0 as the direct limit of its subcategories $\langle K \rangle$. The inverse limit of the corresponding Tannaka groups, with the p_{LK} as transition morphisms, is a pro-algebraic group scheme $G(\mathbf{D})$ over Λ . For $\mathbf{D} = D_c^b(X, \Lambda)$ we put $G(X) = G(\mathbf{D})$.

Lemma 29. *Suppose $k = \mathbb{C}$ and $\Lambda = \mathbb{C}$. Then the maximal abelian quotient group of the group of connected components $\pi_0(G(X))$ is*

$$\pi_0(G(X))^{ab} = \pi_1^{et}(\hat{X}, 0)(-1)$$

where \hat{X} denotes the dual abelian variety of X and where the Tate twist (-1) refers to the $\text{Gal}(F)$ -action for any subfield $F \subseteq \Lambda$ over which X is defined.

Proof. For $K \in \mathbf{P}^0$ the epimorphism $G(K) \twoheadrightarrow \pi_0(G(K))^{ab}$ defines the full subcategory $\text{Rep}_\Lambda(\pi_0(G(K))^{ab}) \subseteq \text{Rep}_\Lambda(G(K)) = \langle K \rangle$ generated by the characters of $G(K)$ of finite order. For $\mathbf{D} = D_c^b(X, \Lambda)$ and $k = \Lambda = \mathbb{C}$, by proposition 21 b), any character of $G(K)$ is represented by a skyscraper sheaf δ_x supported in some point $x \in X(\mathbb{C})$. Since $\delta_x * \delta_y = \delta_{x+y}$, such a character has finite order iff x is a torsion point in $X(\mathbb{C})$. By Pontryagin duality therefore $\pi_0(G(X))^{ab}(1) = \varprojlim_{n \in \mathbb{N}} \text{Hom}(X(\mathbb{C})[n], \mathbb{G}_m)$, which can be identified with $\pi_1^{et}(\hat{X}, 0)$. \square

Lemma 30. *Every homomorphism $f : X \rightarrow Y$ of abelian varieties over k induces a homomorphism of pro-algebraic groups*

$$G(f) : G(Y) \rightarrow G(X) .$$

If f is surjective, then $G(f)$ is a closed embedding.

Proof. One easily checks $Rf_*(K * L) = Rf_*(K) * Rf_*(L)$. Furthermore, we have $Rf_*(\text{Perv}(X, \Lambda)^0) \subseteq \text{Perv}(Y, \Lambda)^0$ and $Rf_*(\mathbf{N}_{coh}) \subseteq \mathbf{N}_{coh}$. Indeed, by dévissage it suffices to check this for *simple* perverse sheaves in the respective categories, and then it follows from the decomposition theorem applied to the proper morphism f . Hence Rf_* induces a tensor functor ACU from $\text{Perv}(X, \Lambda)_{coh}^0$ to $\text{Perv}(Y, \Lambda)_{coh}^0$.

If f is surjective, the morphism $G(f)$ is a closed embedding. Indeed, by [DM, prop. 2.21 b)] it is enough to show that for perverse K on Y there exists a perverse L on X such that K is a retract of $Rf_*(L)$. This assertion

can be reduced to the cases where f is either an isogeny or a projection $X = Z \times Y \rightarrow Y$ onto a factor. In both cases, for $L = f^*(K)$ resp. $L = \mathbf{1} \boxtimes K$, our assertion is evident. \square

12. NEARBY CYCLES

In this section we describe the behaviour of the Tannaka groups $G(X)$ when X varies in an algebraic family. For $i \in \{0, 1\}$, let X_i be an abelian variety over an algebraically closed field k_i which has characteristic zero or is the algebraic closure of a finite field, and put $\Lambda = \mathbb{C}$ or $\Lambda = \overline{\mathbb{Q}}_l$ for some $l \neq \text{char}(k_0), \text{char}(k_1)$. Suppose we have Λ -linear rigid symmetric monoidal categories \mathbf{D}_i with $\text{End}_{\mathbf{D}_i}(\mathbf{1}) \cong \Lambda$ and faithful Λ -linear tensor functors ACU $\text{rat}_i : \mathbf{D}_i \rightarrow D_c^b(X_i, \Lambda)$, and assume that \mathbf{D}_0 and \mathbf{D}_1 satisfy axioms (3), (T) and (S) as in the previous section.

Let $\psi_{\mathbf{D}} : \mathbf{D}_0 \rightarrow \mathbf{D}_1$ and $\psi_D : D_c^b(X_0, \Lambda) \rightarrow D_c^b(X_1, \Lambda)$ be triangulated t -exact tensor functors ACU such that the diagram

$$\begin{array}{ccc} \mathbf{D}_0 & \xrightarrow{\text{rat}} & D_c^b(X_0, \Lambda) \\ \psi_{\mathbf{D}} \downarrow & & \downarrow \psi_D \\ \mathbf{D}_1 & \xrightarrow{\text{rat}} & D_c^b(X_1, \Lambda) \end{array}$$

commutes, with functorial isomorphisms $H^\bullet(X_1, \psi_D(K)) \cong H^\bullet(X_0, K)$ for all $K \in D_c^b(X_0, \Lambda)$. Assume that we can identify $\pi_1(X_0, 0) = \pi_1(X_1, 0)$ such that for all characters χ of this group we have $\psi_{\mathbf{D}}((-)_\chi) = (\psi_D(-))_\chi$. In what follows we simply write ψ for both $\psi_{\mathbf{D}}$ and ψ_D .

Lemma 31. *Let $K \in \mathbf{P}_0$. If $\psi(K)_\chi \in \mathbf{P}_1^0$ for some character χ of $\pi_1(X_1, 0)$, then we have a closed embedding (depending on the choice of χ)*

$$G(\psi(K)) \hookrightarrow G(K).$$

Proof. Since $\psi(K)_\chi = \psi(K_\chi)$, we can assume χ is trivial. By assumption then $\psi(K) \in \mathbf{P}_1^0$, and since ψ is exact and preserves perverse zero types, it follows that $K \in \mathbf{P}_0^0$. Using the description of morphisms in $\langle K \rangle$ in the proof of lemma 27, it follows that ψ induces a tensor functor ACU from $\langle K \rangle$ to $\langle \psi(K) \rangle$. So we can proceed as in the proof of lemma 30. \square

The above result applies in particular in the following situation. Let S be the spectrum of a Henselian discrete valuation ring with closed point s and generic point η . Let $\overline{\eta}$ be a geometric point over η . Let \overline{S} be the normalization of S in the residue field $\kappa(\overline{\eta})$, and let \overline{s} be a geometric point of \overline{S} over s . For an abelian scheme X over S , put $\overline{X} = X \times_S \overline{S}$. Consider the

commutative diagramm

$$\begin{array}{ccccc} X_{\bar{s}} & \xhookrightarrow{\bar{i}} & \bar{X} & \xleftarrow{\bar{j}} & X_{\bar{\eta}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{s} & \hookrightarrow & \bar{S} & \longleftarrow & \bar{\eta} \end{array}$$

where \bar{i} and \bar{j} are the natural morphisms. We then have the functor of nearby cycles [SGA7, exp. XIII-XIV]

$$\psi = \bar{i}^* R\bar{j}_* : \mathbf{D}_0 = D_c^b(X_{\bar{\eta}}, \Lambda) \longrightarrow \mathbf{D}_1 = D_c^b(X_{\bar{s}}, \Lambda).$$

This functor is t -exact for the perverse t -structures by [II, cor. 4.5], and theorem 4.7 in loc. cit. implies that it is a tensor functor for convolution. Note that $H^\bullet(X_{\bar{s}}, \psi(K)) = H^\bullet(X_{\bar{\eta}}, K)$, for all $K \in D_c^b(X_{\bar{\eta}}, \Lambda)$. Furthermore, since \bar{X} is proper and smooth over \bar{S} , by [SGA1, exp. X, cor. 3.9] we have a specialization epimorphism $sp : \pi_1(X_{\bar{\eta}}, 0) \twoheadrightarrow \pi_1(X_{\bar{s}}, 0)$ whose kernel is a pro- p -group for $p = \text{char}(\kappa(\bar{s}))$. If $\kappa(\bar{s})$ has characteristic zero, then sp is an isomorphism; extending local systems on $X_{\bar{\eta}}$ to local systems on \bar{X} , one then sees that $\psi(L_\chi) = L_{\chi \circ sp^{-1}}$ for any character χ of $\pi_1(X_{\bar{\eta}}, 0)$. In this case we also write χ for the character $\chi \circ sp^{-1}$ of $\pi_1(X_{\bar{s}}, 0)$ by abuse of notation; then $\psi(K_\chi) = (\psi(K))_\chi$ for all χ and all $K \in D_c^b(X_{\bar{\eta}}, \Lambda)$.

13. THE SPECTRUM OF A PERVERSE SHEAF

Let X be a complex abelian variety of dimension g . Then $\pi_1(X, 0) \cong \mathbb{Z}^{2g}$ and the group $\Pi(X)$ of characters $\chi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$ is a complex algebraic torus of rank $2g$. For any perverse sheaf P on X , we explain in this section how to determine the set of all $\chi \in \Pi(X)$ for which theorem 2 fails, and in particular we show that this set is a finite union of translates of proper algebraic subtori of $\Pi(X)$. We also consider the corresponding question in the relative setting of theorem 3.

Note that Π is a contravariant functor: Any homomorphism $h : X \rightarrow B$ of abelian varieties induces a homomorphism $\pi_1(h) : \pi_1(X, 0) \rightarrow \pi_1(B, 0)$ and hence a homomorphism $\Pi(h) : \Pi(B) \rightarrow \Pi(X)$ of algebraic tori. For a perverse sheaf P on X we define the spectrum $\mathcal{S}(P) \subset \Pi(X)$ to be the set of all $\chi \in \Pi(X)$ such that

$$H^i(X, P_\chi) \neq 0 \quad \text{for some } i \neq 0.$$

More generally, for a semisimple complex $K = \bigoplus_{n \in \mathbb{Z}} {}^p H^{-n}(K)[n]$ on X we define

$$\mathcal{S}(K) = \bigcup_{n \in \mathbb{Z}} \mathcal{S}({}^p H^{-n}(K)).$$

It follows from the definitions that $\mathcal{S}(K_\chi) = \chi^{-1} \cdot \mathcal{S}(K)$ for all $\chi \in \Pi(X)$ and that for all semisimple K_1, K_2 we have

$$\mathcal{S}(K_1 * K_2) \subseteq \mathcal{S}(K_1) \cup \mathcal{S}(K_2) = \mathcal{S}(K_1 \oplus K_2).$$

In particular, the last equality reduces the computation of the spectrum of semisimple sheaf complexes to the case of simple perverse sheaves. Note that $\mathcal{S}(P)$ may be empty; for example, this is the case if P is a skyscraper sheaf or if $P = i_*E[1]$ where $i : C \hookrightarrow X$ is the embedding of a smooth curve in X and where E is an irreducible local system on C of rank at least two.

Remark 32. *The functor Π has the following properties.*

- (1) *Let $g : X \rightarrow B$ be an isogeny with kernel F . Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Hom}(F, \mathbb{C}^*) \longrightarrow \Pi(B) \xrightarrow{\Pi(g)} \Pi(X) \longrightarrow 0.$$

For any perverse sheaf P on X the direct image $g_(P)$ is a perverse sheaf on B , and $\Pi(g)$ induces a surjection*

$$\mathcal{S}(g_*(P)) \twoheadrightarrow \mathcal{S}(P).$$

- (2) *Let $i : A \hookrightarrow X$ be the inclusion of an abelian subvariety with quotient morphism $q : X \rightarrow B = X/A$. Then we have an exact sequence*

$$0 \longrightarrow \Pi(B) \xrightarrow{\Pi(q)} \Pi(X) \xrightarrow{\Pi(i)} \Pi(A) \longrightarrow 0.$$

In this situation we denote by $K(A) \subseteq \Pi(X)$ the image of $\Pi(q)$.

Proof. The exactness of the considered sequences can be seen from the description of a complex abelian variety as the quotient of a complex vector space modulo a lattice. For the surjectivity $\mathcal{S}(g_*(P)) \twoheadrightarrow \mathcal{S}(P)$ in part (1) use that $H^i(X, P_{\Pi(g)(\chi)}) = H^i(B, g_*(P)_\chi)$ and that $\Pi(g)$ is surjective. \square

In what follows, we denote by $E(X)$ the class of all semisimple perverse sheaves on X with Euler characteristic zero. A perverse sheaf will be called clean if it does not contain constituents from $E(X)$. For $x \in X(\mathbb{C})$ we denote by $t_x : X \rightarrow X$ the translation morphism $y \mapsto x + y$, and for $K \in D_c^b(X, \mathbb{C})$ we consider the stabilizer

$$\mathrm{Stab}(K) = \{x \in X(\mathbb{C}) \mid t_x^*(K) \cong K\}.$$

Its connected component $\mathrm{Stab}(K)^0 \subseteq \mathrm{Stab}(K)$ is an abelian subvariety of X .

Lemma 33. *Let $P \in E(X)$ be a simple perverse sheaf, and consider the algebraic subtorus $K(A) \subset \Pi(X)$ from remark 32 with $A = \mathrm{Stab}(K)^0$. This is a proper subtorus, and there exist $\chi_1, \dots, \chi_n \in \Pi(X)$ such that*

$$\mathcal{S}(P) = \bigcup_{i=1}^n \chi_i^{-1} \cdot K(A).$$

Furthermore, a character χ lies in $\mathcal{S}(P)$ if and only if $H^\bullet(X, P_\chi) \neq 0$.

Proof. By proposition 21 we have $\dim(A) > 0$, and there exists an abelian variety B and an isogeny $f : A \times B \rightarrow X$ such that $f^*(P) = \delta_\psi \boxtimes Q$ for some character $\psi \in \Pi(A)$ with associated perverse sheaf $\delta_\psi = L_\psi[\dim(A)]$ and for

some clean perverse sheaf Q on B . Choose an isogeny $g : X \rightarrow A \times B$ such that $g \circ f = n \cdot \text{id}$ for some $n \in \mathbb{N}$. By adjunction and by semisimplicity, P is a direct summand of $f_* f^*(P) = f_*(\delta_\psi \boxtimes Q)$, so $g_*(P)$ is a direct summand of $g_* f_*(\delta_\psi \boxtimes Q) = n_* \delta_\psi \boxtimes n_* Q$. Here $n_* \delta_\psi$ decomposes into a direct sum of local systems, and $n_* Q$ is a clean perverse sheaf by adjunction since Q is clean. Altogether this shows

$$g_*(P) = \bigoplus_{i=1}^n \delta_{\chi_i} \boxtimes Q_i$$

for certain $\chi_i \in \Pi(A)$ and certain semisimple clean perverse sheaves Q_i on B . Now for any $(\chi, \varphi) \in \Pi(A \times B) = \Pi(A) \times \Pi(B)$ we have

$$H^\bullet(A \times B, (\delta_{\chi_i} \boxtimes Q_i)_{(\chi, \varphi)}) = H^\bullet(A, \delta_{\chi_i \chi}) \otimes H^\bullet(B, (Q_i)_\varphi),$$

where $H^\bullet(B, (Q_i)_\varphi) \neq 0$ because $Q_i \notin E(B)$. On the other hand,

$$H^\bullet(A, \delta_{\chi_i \chi}) = \begin{cases} 0 & \text{for } \chi \neq \chi_i^{-1}, \\ H^\bullet(A, \mathbb{C}) & \text{for } \chi = \chi_i^{-1}. \end{cases}$$

So $\mathcal{S}(g_*(P)) = \bigcup_{i=1}^n \{\chi_i^{-1}\} \times \Pi(B)$, and we are done by remark 32. \square

Corollary 34. *For any semisimple perverse sheaf P on X there are non-zero abelian subvarieties $A_1, \dots, A_n \subseteq X$ and $\chi_1, \dots, \chi_n \in \Pi(X)$ such that*

$$\mathcal{S}(P) = \bigcup_{i=1}^n \chi_i \cdot K(A_i).$$

Proof. By theorem 20 applied to the class $\mathbf{N} = N_{Euler}$ of complexes with perverse cohomology sheaves in $E(X)$, we have

$$P^{*g} = Q \oplus \bigoplus_v N_v[v]$$

where Q is a clean semisimple perverse sheaf and the N_v are semisimple perverse sheaves in $E(X)$. Since twisting with a character is a tensor functor by proposition 9, it follows for any $\chi \in \Pi(X)$ that

$$(13.1) \quad H^\bullet(X, P_\chi)^{\otimes g} = H^\bullet(X, Q_\chi) \oplus \bigoplus_v H^\bullet(X, (N_v)_\chi)[v].$$

We claim that $\mathcal{S}(P) = \bigcup_v \mathcal{S}(N_v)$; by lemma 33 this suffices to finish the proof. Indeed, suppose we are given $\chi \in \mathcal{S}(P)$. Then $H^\bullet(X, P_\chi)$ is not concentrated in degree zero, hence (13.1) is non-zero in some degree d with $|d| \geq g$. Since Q is a clean perverse sheaf, $H^\bullet(X, Q_\chi)$ vanishes in all such degrees d , so $H^\bullet(X, (N_v)_\chi) \neq 0$ for some v and hence $\chi \in \mathcal{S}(N_v)$ by the last part of lemma 33. Conversely, if $\chi \in \mathcal{S}(N_v)$ for some v , then $H^\bullet(X, (N_v)_\chi)$ is non-zero in at least two different cohomology degrees; then by (13.1) the same holds for $H^\bullet(X, P_\chi)$, so $\chi \in \mathcal{S}(P)$. \square

For a homomorphism $f : X \rightarrow B$ of abelian varieties, define the relative spectrum $\mathcal{S}_f(P)$ of a perverse sheaf P on X to be the set of all $\chi \in \Pi(X)$

such that the direct image $Rf_*(K_\chi)$ is not perverse. By abuse of notation, for $\chi \in \Pi(X)$ and $\psi \in \Pi(B)$ we write $\chi\psi = \chi \cdot (\Pi(f)(\psi)) \in \Pi(X)$. Then the projection formula shows

$$Rf_*(P_{\chi\psi}) = (Rf_*(P_\chi))_\psi,$$

hence $\mathcal{S}_f(P)$ is invariant under $\Pi(B)$. In particular, if $B = X/A$ for an abelian subvariety $A \subseteq X$, then $\mathcal{S}_f(P)$ is determined by its image $\overline{\mathcal{S}}_f(P)$ in $\Pi(A) = \Pi(X)/\Pi(B)$. Furthermore, in theorem 3 the assertion for *most characters* can be read in $\Pi(A)$, i.e. theorem 3 holds in the stronger sense that $\overline{\mathcal{S}}_f(P)$ is contained in a finite union of translates of proper algebraic subtori of $\Pi(A)$. In fact we have the following

Lemma 35. $\mathcal{S}_f(P) \subseteq \mathcal{S}(P) \cdot \Pi(B)$.

Proof. If $\chi \in \Pi(X) \setminus (\mathcal{S}(P) \cdot \Pi(B))$, then $\chi\psi \notin \mathcal{S}(P)$ for all $\psi \in \Pi(B)$. So $H^\bullet(X, P_{\chi\psi}) = H^\bullet(B, (Rf_*(P_\chi))_\psi)$ is not concentrated in degree zero for any $\psi \in \Pi(B)$. By theorem 2 then the complex $Rf_*(P_\chi)$ is not perverse, so we have $\chi \notin \mathcal{S}_f(P)$. \square

The complement $\mathcal{S}(P) \setminus \mathcal{S}_f(P)$ can be described in the following way. A character χ lies in this complement if and only if $Rf_*(P_\chi)$ is perverse but $H^i(B, Rf_*(P_\chi)) = H^i(X, P_\chi) \neq 0$ for some $i \neq 0$, hence

$$\mathcal{S}(P) \setminus \mathcal{S}_f(P) = \bigcup_{\chi \in \mathcal{S}(P) \setminus \mathcal{S}_f(P)} \chi \cdot \mathcal{S}(Rf_*(P_\chi))$$

where by our usual convention $\mathcal{S}(Rf_*(P_\chi))$ is viewed as a subset of $\Pi(X)$.

14. APPENDIX: REDUCTIVE SUPERGROUPS

In this appendix we recall the definition of an algebraic supergroup and collect some basic facts about these in the reductive case. Throughout let Λ be an algebraically closed field of characteristic zero. As in [We3, p. 16] we consider triples $\mathbf{G} = (G, \mathfrak{g}_-, Q)$ consisting of

- a classical algebraic group G over Λ , whose Lie algebra with the adjoint action of G we denote by $\mathfrak{g}_+ = \text{Lie}(G)$,
- a finite-dimensional algebraic representation \mathfrak{g}_- of G over Λ , given by a homomorphism $Ad_- : G \rightarrow \text{Gl}(\mathfrak{g}_-)$,
- a G -equivariant quadratic form $Q : \mathfrak{g}_- \rightarrow \mathfrak{g}_+$, $Q(v) = [v, v]$ defined by a symmetric Λ -bilinear form $[\cdot, \cdot] : \mathfrak{g}_- \times \mathfrak{g}_- \rightarrow \mathfrak{g}_+$.

Such a triple \mathbf{G} is called an algebraic supergroup over Λ , if the differential $ad_- = \text{Lie}(Ad_-)$ of Ad_- satisfies $ad_-(Q(v))(v) = 0$ for all $v \in \mathfrak{g}_-$. We define a homomorphism

$$h : (G_1, \mathfrak{g}_{1-}, Q_1) \longrightarrow (G_2, \mathfrak{g}_{2-}, Q_2)$$

of algebraic supergroups over Λ to be a pair $h = (f, g)$, where $f : G_1 \rightarrow G_2$ is a homomorphism of algebraic groups and $g : \mathfrak{g}_{1-} \rightarrow \mathfrak{g}_{2-}$ a Λ -linear and f -equivariant map such that $Q_2 \circ g = \text{Lie}(f) \circ Q_1$. Such a homomorphism is a mono- resp. an epimorphism of algebraic supergroups iff both f and

g are mono- resp. epimorphisms. We define the parity automorphism of an algebraic supergroup $\mathbf{G} = (G, \mathfrak{g}_-, Q)$ to be $h = (id_G, -id_{\mathfrak{g}_-}) : \mathbf{G} \rightarrow \mathbf{G}$. These constructions are motivated by the following example.

Let $A = A_+ \oplus A_-$ be an affine super Hopf algebra over Λ , i.e. a graded commutative $\mathbb{Z}/2\mathbb{Z}$ -graded Hopf algebra of finite type over Λ . Let $J \trianglelefteq A$ be the ideal generated by A_- . Then $G = Spec A/J$ is a classical algebraic group, and the left invariant super derivations of A form a super Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ with a natural action of G extending the adjoint action on $\mathfrak{g}_+ = Lie(G)$. If we take $Q(v) = [v, v]$ for the super bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, then by loc. cit. $\mathbf{G} = (G, \mathfrak{g}_-, Q)$ is an algebraic supergroup over Λ . By loc. cit. this realizes the opposite of the category of affine super Hopf algebras as a full subcategory of the category of algebraic supergroups. Hence for algebraic supergroups associated to affine super Hopf algebras, the notions introduced here are compatible with those in [De2].

A particular instance is the general linear supergroup $\mathbf{G} = \mathbf{GL}(V)$ attached to a super vector space $V = V_+ \oplus V_-$ of finite dimension over Λ . In this case $G = GL(V_+) \times GL(V_-)$, $\mathfrak{g}_- = Hom_{\Lambda}(V_+, V_-) \oplus Hom_{\Lambda}(V_-, V_+) \subset End_{\Lambda}(V)$ with the adjoint action of G , and one takes $Q(A \oplus B) = AB + BA$.

For any algebraic supergroup $\mathbf{G} = (G, \mathfrak{g}_-, Q)$ over Λ , let $\mathbf{G}^0 = (G^0, \mathfrak{g}_-, Q)$ denote its Zariski connected component, and define its supercenter to be $Z(\mathbf{G}) = (Z, 0, 0)$ where $Z \subseteq Z(G)$ is the largest central subgroup of G acting trivially on \mathfrak{g}_- . For $g \in G$ we put $int(g) = (g^{-1}(-)g, Ad_-(g)) : \mathbf{G} \rightarrow \mathbf{G}$. Then $Z \subseteq G$ is the subgroup of all $g \in G$ such that $int(g) = id_{\mathbf{G}}$.

A super representation of \mathbf{G} is a homomorphism $\rho_V : \mathbf{G} \rightarrow \mathbf{GL}(V)$ for some super vector space V . By definition, a homomorphism between two super representations ρ_V and ρ_W is a homomorphism $V \rightarrow W$ of super vector spaces such that the induced homomorphism $h : \mathbf{GL}(V) \rightarrow \mathbf{GL}(W)$ satisfies $h \circ \rho_V = \rho_W$. The category $Rep_{\Lambda}(\mathbf{G})$ of super representations of \mathbf{G} over Λ is an abelian Λ -linear rigid symmetric monoidal category with respect to the super tensor product. We also have Schur's lemma:

Lemma 36. *Let $\rho_V : \mathbf{G} \rightarrow \mathbf{GL}(V)$ be an irreducible super representation. Then every endomorphism ϕ of ρ_V has the form $\phi = \lambda \cdot id_V$ for some $\lambda \in \Lambda$.*

Proof. The proof works as in the classical case. Notice that by definition we only consider endomorphisms preserving the $\mathbb{Z}/2\mathbb{Z}$ -grading. Otherwise Schur's lemma would have to be modified, see [Sc, prop. 2, p. 46]. \square

In particular, it follows that the super center $Z(\mathbf{G})$ acts on any irreducible super representation of \mathbf{G} by a character $\chi : Z(\mathbf{G}) \rightarrow \Lambda^*$ (recall that the super center is a classical commutative algebraic group and that each of its elements defines an endomorphism of any given super representation of \mathbf{G}).

An algebraic supergroup $\mathbf{G} = (G, \mathfrak{g}_-, Q)$ is called *reductive*, if the abelian category $Rep_{\Lambda}(\mathbf{G})$ is semisimple. Let us briefly recall the classification of reductive supergroups from [We3]. Every classical reductive group G ,

considered as a supergroup with $\mathfrak{g}_- = 0$, is a reductive supergroup. Other examples include the orthosymplectic supergroups $\mathbf{Spo}_\Lambda(2r, 1)$ with $r \in \mathbb{N}$, defined as follows: Fix a non-degenerate antisymmetric $2r \times 2r$ matrix J over Λ , and consider $\mathrm{Sp}_\Lambda(2r, J) = \{g \in \mathrm{Gl}_\Lambda(2r) \mid g^t J g = J\}$. Then

$$\mathbf{Spo}_\Lambda(2r, 1) = (\mathrm{Sp}_\Lambda(2r, J), \Lambda^{2r}, Q),$$

where Λ^{2r} is equipped with the standard action of $\mathrm{Sp}_\Lambda(2r, J)$ and where the map $Q : \Lambda^{2r} \rightarrow \mathrm{Sp}_\Lambda(2r, J)$ is defined by $Q(v)_{ik} = \sum_{j=1}^{2r} v_i v_j J_{jk}$. A different choice of the matrix J gives an isomorphic supergroup.

In general, by theorem 6 of loc. cit., an algebraic supergroup \mathbf{G} over Λ is reductive iff there exists a classical reductive group H and $N \in \mathbb{N}_0$, $n_i, r_i \in \mathbb{N}$ such that \mathbf{G} is isomorphic to a semidirect product

$$\mathbf{G} = \left(\prod_{i=1}^N (\mathbf{Spo}_\Lambda(2r_i, 1))^{n_i} \right) \rtimes H$$

defined by a homomorphism $\pi_0(H) \rightarrow \prod_{i=1}^N \mathfrak{S}_{n_i}$ where each symmetric group \mathfrak{S}_{n_i} acts on $(\mathbf{Spo}_\Lambda(2r_i, 1))^{n_i}$ by permutation of the factors.

Corollary 37. *Let \mathbf{G} be a reductive supergroup over Λ . Then the underlying classical group G is reductive, and the super center $Z(\mathbf{G})$ is a subgroup of finite index in the center $Z(G)$.*

Proof. By the above it suffices to show this if $\mathbf{G} = \mathbf{Spo}_\Lambda(2r, 1)$ for some $r \in \mathbb{N}$. But then $G = \mathrm{Sp}_\Lambda(2r)$, and $Z(G) = \mu_2$ is finite. \square

Proposition 38. *Let $h : \mathbf{G}_1 \rightarrow \mathbf{G}_0$ be a homomorphism between reductive supergroups over Λ which induces an epimorphism $f : G_1 \twoheadrightarrow G_0$ on the underlying classical groups. If the super center $Z(\mathbf{G}_0)$ contains a classical torus T_0 , then $Z(\mathbf{G}_1)$ contains a classical torus T_1 such that the induced map $p : T_1 \rightarrow T_0$ is an isogeny.*

Proof. The category of tori (or diagonalizable groups) over Λ , up to isogeny, is the equivalent to the category of finite vector spaces over \mathbb{Q} via the cocharacter functor $T \mapsto X(T) = \mathrm{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{Q}$. If π is a finite group acting on T , then $X((T^\pi)^0) = X(T)^\pi$ for the fixed group T^π .

For reductive supergroups \mathbf{G} note $Z(\mathbf{G})^0 = Z(G)^0$ by corollary 37. On $Z(G^0)$ the group G acts by conjugation, thus defines an action of the finite group $\pi = \pi_0(G)$. By definition $Z(G)^0 \subset Z(G^0)^\pi$ and this is a subgroup of finite index: $Z(G)^0 = (Z(G^0)^\pi)^0$. This follows by an application of the cocharacter functor since $X(Z(G)^0) = X(Z(G^0)^\pi)$.

For the proof of the proposition it suffices to show that T_0 is contained in the image of $Z(G_1)^0$. By assumption h induces an epimorphism of the underlying reductive classical groups $f : G_1 \rightarrow G_0$, hence an epimorphism $f : (G_1)^0 \rightarrow (G_0)^0$ of their connected components. By standard properties of classical connected reductive groups $T = Z((G_0)^0)^0$ is the image of the

torus $S = Z((G_1)^0)^0$. The epimorphism $f : S \rightarrow T$ is equivariant for the action of $\pi = \pi_0(G_1)$ on S and T , where the latter is induced by the natural homomorphism $\pi_0(G_1) \rightarrow \pi_0(G_0)$. We claim that $h : (S^\pi)^0 \rightarrow (T^\pi)^0$ is surjective. Indeed, the functor of invariants under a finite group π is right exact on the category of finite-dimensional vector spaces over \mathbb{Q} . Since $(S^\pi)^0 \subset Z(G_1)^0$ and $T_0 \subset (T^\pi)^0$ this completes the proof. \square

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MATHEMATISCHES INSTITUT, RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG, IM
NEUENHEIMER FELD 288, 69120 HEIDELBERG, GERMANY

E-mail address: tkraemer@mathi.uni-heidelberg.de

E-mail address: weissauer@mathi.uni-heidelberg.de